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NEWS OF HIGHER EDUCATIONAL INSTITUTIONS,

MINISTRY OF HIGHER EDUCATION, USSR,

RADIOPHYSICS SERIES

[Following is a complete translation of the publication entitled "Izvestiya Vysshikh Uchebnykh Zavedeniy MVO SSSR po Razdelu Radiofizika" (English version above), Vol. 2, No 6, 1959, pages 843-1017.]

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THEORY OF FORMATION OF IONOSPHERIC IRREGULARITIES IN THE F LAYER

(This is a translation of an article written by B. N. Gershman and V. P. Dokuchaev in Radiofizika (Radiophysics), Vol. II, No. 6, 1959, pages 843-847.)

(abstract) Calculations are carried out which make it possible in a quantitative way to estimate the effectiveness of the mechanism for the formation of inhomogeneities in the F-layer advanced by Martyn and Dagg. It is shown that this mechanism can explain the formation of more or less regular motion in the F-layer; however, it is not in a position to account for the formation in this layer of large scale irregularities of the order of 2-4 kilometers.

The study of ionospheric irregularities occupies at the present time an important place in radio investigations of the ionosphere. Along with the numerous experimental investigations, a number of mechanisms have been proposed for the formation of irregularities in the electronic concentration in the ionosphere. A listing and a critical discussion of these mechanisms can be found in Refs. 1-3. Analysis of these researches shows that the greatest difficulty lies in the theoretical interpretation of the irregularities arising in the F-layer.

Here we shall limit ourselves only to one of the mechanisms of formation of irregularities. This mechanism was proposed by Martyn [4]; later, its possibilities were analyzed in detail by Dagg [5]. According to the Martyn-Dagg mechanism, the appearance of irregularities in the F-layer is connected with the transfer of electric fields into this layer from the dynamic region. (The dynamic region lies at an altitude of $z \sim 130$ km.) Irregular components of these fields lead to a drift of charged particles in the F-layer, with which is associated the ionospheric "wind" observed in experiment. Varying of the component of the transferrable electric field should lead to the appearance of ionospheric irregularities.

In the papers of Martyn and Dagg just mentioned, there is fundamentally contained only the formulation of a suggested

hypothesis, and quantitative estimates are completely lacking. Furthermore, calculations make it possible to demonstrate more clearly the possibilities of the given mechanism, as is evident from what follows. Below, we shall carry out a calculation of the effect of leakage of the electric field from the E-layer to the F-layer, and shall discuss the formulas that are obtained. We shall show that the Martyn-Dagg mechanism by itself is insufficient to explain the experimental data.

1. In a consideration of the leakage of the field from the E-layer to the F-layer, we come to the problem of the skin effect in plasma. In this case, account of the anisotropy of the conductivity which is connected with the magnetic field of the earth H_0 is shown to be very important, as is also the change in the conductivity with altitude.

One can start out directly from the microscopic equation for the electric field E:

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\,\partial j/\partial t, \tag{1}$$

where j is the total microscopic current. This equation is written for the case in which $\nabla E = 4\pi e(N_i - N_e) \simeq 0$ ($-e$ is the charge on the electron, N_e and N_i are the concentrations of electrons and ions). The latter condition is satisfied for quasi-neutral plasma, if $N_e \simeq N_i$. It should be realized for not very fast, quasi-stationary motions in the ionosphere. We shall have to deal with precisely this type of motion.

The current j is connected with the electric field E by the generalized law of Ohm, which, for quasi-static processes can be written in the form:

$$j = \sigma_0 (hE')h + \sigma_1 [[hE']h] + \sigma_2 [hE'], \tag{2}$$

where h is a unit vector in the direction of the field H_0. Here, because of the presence in the general case of motion of the medium, the field E does not appear in (2), but the vector E', which is defined by the relation

$$E' = E + E_d = E + \frac{1}{c}[vH_0], \tag{3}$$

where the field $E_d = \frac{1}{c}[vH_0]$ is known as the dynamic field (v is the velocity of the medium). In (2), σ_0, σ_1 and σ_2 are, respectively, the longitudinal and transverse conductivities and the Hall conductivity:

$$\sigma_0 \simeq \frac{e^2 N}{m\nu_e}; \quad \sigma_1 = e^2 N\left[\frac{\nu_e}{m(\omega_H^2+\nu_e^2)} + \frac{\nu_i}{M(\Omega_H^2+\nu_i^2)}\right];$$

$$\sigma_2 = e^2 N\left[\frac{\omega_H}{m(\omega_H^2+\nu_e^2)} - \frac{\Omega_H}{M(\Omega_H^2+\nu_i^2)}\right]. \tag{4}$$

Here ν_e is the effective number of collisions of electrons with other particles, ν_i is the number of collisions for ions, ω_H and Ω_H are the gyrofrequencies for electrons and ions, m and M are the masses of electrons and ions, $N \equiv N_e \simeq N_i$.

Keeping in mind a definite point of the earth's sphere, we select the following set of Cartesian coordinates. The x axis is directed along the geomagnetic equator, the y axis to the east, the z axis vertically upward. In these coordinates, the earth's magnetic field $\underline{H}_0 = H_0 \underline{h} = -H_0 \cos\chi\, \underline{i} - H_0 \sin\chi\, \underline{k}$, where χ is the magnetic declination. Writing Eq. (1) in coordinate form, and taking Eq. (2) into account in this case, we obtain:

$$\nabla^2 E_x - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\{(\sigma_0\cos^2\chi + \sigma_1\sin^2\chi)E'_x + \sigma_2\sin\chi\, E'_y + (\sigma_0 - \sigma_1)\sin\chi\cos\chi\, E'_z\}; \tag{5}$$

$$\nabla^2 E_y - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\{\sigma_1 E'_y - \sigma_2\sin\chi\, E'_x + \sigma_2\cos\chi\, E'_z\}; \tag{6}$$

$$\nabla^2 E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial}{\partial t}\{(\sigma_0 - \sigma_1)\cos\chi\sin\chi\, E'_x - \sigma_2\cos\chi\, E'_y + (\sigma_0\sin^2\chi + \sigma_1\cos^2\chi)E'_z\}. \tag{7}$$

We now take into account the conditions which hold in the ionosphere at altitudes exceeding 130 km. For the condition $\nu_i > \Omega_H$, the conductivity $\sigma_1 \gg \sigma_2$, so that one can assume approximately that $\sigma_2 \simeq 0$. The change of the conductivities σ_0 and σ_1 with altitude is determined (in terms of the number of collisions) first of all by the change in the concentration of neutral particles, and not by the change of the electronic concentration N with altitude. At altitudes above 130 km, we can assume approximately that

$$\sigma_0 = \sigma_{00} e^{z/z_0}; \quad \sigma_1 = \sigma_{10} e^{-z/z_0},$$

where z_0 is the scale of a regular atmosphere.

After neglecting terms for σ_2 and substitution of the field $\underline{E}'$

for the field $\underset{\sim}{E}^*$, we obtain a set of equations similar to (5)-(7).

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*In substituting $\underset{\sim}{E}'$ for $\underset{\sim}{E}$, we neglect the dynamic field $\underset{\sim}{E}_d$. It is assumed that the dynamic field is large only in the E-layer, in the layer of production of the transferrable fields.

However, even after carrying out these approximations, the solution of the coupled equations for the components of the field E_x and E_z is very complicated (the equation for the component E_y at $\sigma_z = 0$ is split off). Therefore, we shall limit ourselves to two special cases, namely we shall consider leakage at high latitudes ($\chi \cong 90°$, $\cos\chi \ll 1$) and to equatorial regions ($\chi \cong 0$, $\sin\chi \ll 1$). Under these conditions we have, respectively:

$$\nabla^2 E_{x,y} - \frac{1}{c^2}\frac{\partial^2 E_{x,y}}{\partial t^2} = \frac{4\pi\sigma_1}{c^2}\frac{\partial E_{x,y}}{\partial t}; \qquad \nabla^2 E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = \frac{4\pi\sigma_0}{c^2}\frac{\partial E_z}{\partial t}; \qquad (9)$$

$$\nabla^2 E_x - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{4\pi\sigma_0}{c^2}\frac{\partial E_x}{\partial t}; \qquad \nabla^2 E_{y,z} - \frac{1}{c^2}\frac{\partial^2 E_{y,z}}{\partial t^2} = \frac{4\pi\sigma_1}{c^2}\frac{\partial E_{y,z}}{\partial t}. \qquad (10)$$

It follows from (9)-(10) that in these special cases (for the component of the field E_y in the general case) solution of two different types of equations is necessary:

$$\nabla^2 E_\parallel = \frac{4\pi\sigma_{00}}{c^2} e^{z/z_0} \frac{\partial E_\parallel}{\partial t} + \frac{1}{c^2}\frac{\partial^2 E_\parallel}{\partial t^2}; \qquad (11)$$

$$\nabla^2 E_\perp = \frac{4\pi\sigma_{\perp 0}}{c^2} e^{-z/z_0} \frac{\partial E_\perp}{\partial t} + \frac{1}{c^2}\frac{\partial^2 E_\perp}{\partial t^2}. \qquad (12)$$

We shall seek a solution in the form

$$\underset{\sim}{E} \sim E(z) e^{i(\omega t - kx)}, \qquad (13)$$

where k in the general case can be complex. It is evident that by assuming the conductivities to be independent of the horizontal coordinates, we would arrive at the same results even for excitations connected with changes in other directions. Therefore, establishment of the field in the form (13) does not limit the generality of the consideration. Substituting (13) in (11)-(12), we get

$$\frac{d^2 E_\parallel}{dz^2} + \left(\frac{\omega^2}{c^2} - k^2 - i\frac{4\pi\omega\sigma_{00}}{c^2} e^{z/z_0}\right) E_\perp = 0; \qquad (14)$$

$$\frac{d^2 E_\perp}{dz^2} + \left(\frac{\omega^2}{c^2} - k^2 - i\frac{4\pi\omega\sigma_{\perp 0}}{c^2} e^{z z_0}\right) E_\perp = 0. \qquad (15)$$

Solution of these equations can be obtained by employing the substitution $x = \exp(z/z_0)$ for (14), and $x = \exp(-z/z_0)$ for (15). As a result we get

$$E_{\parallel}(z) = A_{\parallel} H_\nu^{(1)}\left(\frac{2z_0}{\lambda_{00}} e^{i3\pi/4 + z/2z_0}\right) + B_{\parallel} H_\nu^{(2)}\left(\frac{2z_0}{\lambda_{00}} e^{i3\pi/4 + z/2z_0}\right); \quad (16)$$

$$E_{\parallel}(z) = A_{\perp} H_\nu^{(1)}\left(\frac{2z_0}{\lambda_{\perp 0}} e^{i3\pi/4 - z/2z_0}\right) + B_{\perp} H_\nu^{(2)}\left(\frac{2z_0}{\lambda_{\perp 0}} e^{i3\pi/4 - z/2z_0}\right), \quad (17)$$

where $H_\nu^{(1,2)}$ are the Hankel functions of first and second kind of νth order,

$$\nu = 2z_0\sqrt{k^2 - k_0^2}; \quad k_0 = \omega/c;$$

$$\lambda_{00} = c/\sqrt{4\pi\omega\sigma_{00}}; \quad \lambda_{\perp 0} = c/\sqrt{4\pi\omega\sigma_{\perp 0}}.$$

The quantities λ_{00} and $\lambda_{\perp 0}$ determine the thickness of the skin layer upon the appearance of an electric field parallel to the magnetic field H_0 or perpendicular to it, respectively.

For the determination of the integration constantsin (16) and (17) it is first of all necessary to take it into account that in a medium with absorption the fields must vanish as the coordinate $z \to \infty$. We then obtain that $B_{\parallel} = 0$, $A_{\perp} = B_{\perp}$ and, consequently, the solutions of (16)-(17) of interest to us can be written in the form

$$E_{\parallel}(z) = A_{\parallel} H_\nu^{(1)}\left[\frac{2z_0}{\lambda_{00}} e^{i3\pi/4 + z/2z_0}\right]; \quad E_{\perp}(z) = 2A_{\perp} J_\nu\left[\frac{2z_0}{\lambda_{\perp 0}} e^{i3\pi/4 - z/2z_0}\right].$$

We shall further assume that the generation of fields takes place in the E-layer. This generation is not accompanied by any sort of discontinuities or very rapid changes in the properties of the medium. Then $A_{\parallel}$ and $A_{\perp}$ are determined from the conditions that $E_{\perp}(0) = E_{\perp 0}$ and $E_{\parallel}(0) = E_{\parallel 0}$ for $z = 0$. As a result, we obtain

$$E_{\parallel}(z) = E_{\parallel 0} H_\nu^{(1)}\left(\frac{2z_0}{\lambda_{00}} e^{i3\pi/4 + z/2z_0}\right) \Big/ H_\nu^{(1)}\left(\frac{2z_0}{\lambda_{00}} e^{i3\pi/4}\right); \quad (18)$$

$$E_{\perp}(z) = E_{\perp 0} J_\nu\left(\frac{2z_0}{\lambda_{\perp 0}} e^{i3\pi/4 - z/2z_0}\right) \Big/ J_\nu\left(\frac{2z_0}{\lambda_{\perp 0}} e^{i3\pi/4}\right). \quad (19)$$

2. We apply Eqs. (18) and (19) for the determination of the effectiveness of the leakage of the components of the electric field $E_{\parallel}$ and $E_{\perp}$, directed along the constant magnetic field H_0 and perpendicularly to it, respectively.

From the sense of the solutions considered, it follows that λ_{00} and $\lambda_{\perp 0}$ are determined by the values of the conductivities σ_{00} and

$\sigma_{\perp 0}$ in the region of the appearance of the field (in the dynamic region). In accordance with the data given in Ref. 6, at an altitude of 130 km, $\lambda_{\perp 0}^{2} = 2 \times 10^{14}\omega^{-1}$, while $\lambda_{00}^{2} = 10^{13}\omega^{-1}$. We take $z_0 = 15$ km for the height of the homogeneous atmosphere z_0.

We shall be interested in processes the periods of change of which lie in the region from tens of seconds to several hours, i.e., processes with circular frequency $\omega \sim 1\text{-}10^{-4}\ \text{sec}^{-1}$.

It is easy to establish the fact that the argument of the Bessel functions in both the numerator and denominator of Eq. (19) is small in magnitude by virtue of the condition $\lambda_{\perp 0} \gg z_0$, and also $\exp(-z/2z_0) \ll 1$. The latter condition is applied because we are interested in the penetration of fields to significant distances (into the region of the maximum of the F-layer). Then, making use of the asymptotic approximations of cylindrical functions, we get from (19):

$$E_\perp(z) \simeq E_{\perp 0}\, e^{-\nu z/2z_0} - E_{\perp 0}\, e^{-\sqrt{k^2 - k_0^2}\, z} \simeq E_{\perp 0}\, e^{-kz}. \qquad (20)$$

The latter equality is valid since the quantity k_0 is very small at the frequencies ω under consideration in comparison with the value of k. We recall that the reciprocal length k determines the character of the irregular structure of the initial distribution of electric fields in the horizontal direction. Effective penetration is possible only to distances $\Delta z \sim 1/k$. At the same time $k \sim 1/\ell$, where ℓ is a horizontal dimension of the irregularities. Thus, penetration of fine scale perturbations with dimensions of the order $\ell \sim 1\text{-}10$ km is shown to be impossible. At the same time, penetration of perturbations with large scales ($\ell \sim 100$ km) from the E-layer to the F-layer is shown to be possible. These fields can lead to the appearance in the F-layer of drift of more or less regular character.

We turn to the determination of the effeciency of penetration of the $E_\parallel$ component. Here we assume that $z \gg z_0$, and also take it into account that $\lambda_{00} > z_0$. Making use of the asymptotic representation for the Hankel function, we get from (18):

$$E_\parallel(z) \simeq E_{\parallel 0} \exp\left[-z/4z_0 - (1+i)\, z_0 \sqrt{2}\, \lambda_{00}^{-1} \exp(z/2z_0)\right]. \qquad (21)$$

It follows from this relation that the penetration of the field from the E-layer to the F-layer at $z \gg z_0$ is practically impossible. In this connection, we not that the decrease of the field in leakage by several fold leads into significant difficulties in comparison with experiment. The fact is that in the decrease of the field one could expect a decrease in the drift velocity in the F-layer. In experiment, the opposite effect is quickly noted: velocities of motion in the F-layer are

higher than velocities of motion in the E-layer.

Thus we come to the conclusion that the transfer is possible only of very large scale irregularities of the electric field. The very great scales correspond to processes of comparatively low frequency. The periods of change of the quantities connected with these processes in every case are not smaller than several minutes. Here only components of the field $\mathbf{E}$ oriented perpendicular to the field $\mathbf{H}_0$ are transferred. Turning to the problem of the nature of the penetration of the irregularities in the F-layer, we are confronted with the necessity of the explanation of the small scale part of these irregularities ($\ell \sim 2$-5 km) due to processes directly in the F-layer itself. One of the possible processes could be convective instability in the region of the F-layer [7]. Recently, it was shown by one of the authors (V. P. Dokuchaev) that the appearance of negative temperature gradients necessary for the convection could be connected with the effect of a current of neutral partibles incident on the earth from interplanetary space. Here, however, estimates of the intensity of the required currents have led to values which are large in comparison with those accepted at the present time.

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THE EXPERIMENTAL STUDY OF PHASE FLUCTUATIONS OF CENTIMETER RADIO WAVES PROPAGATED OVER THE SEA SURFACE

[This is a translation of an article written by A. V. Men', S. Ya. Braude and V. I. Gorbach in Radiofizika (Radiophysics) Vol. II, No. 6, 1959, pages 848-857.]

(Abstract) Results are described of an experimental study of the phase fluctuations of radio waves of ten centimeter range over the sea. A classification is given of the results of observations as a function of the different characteristics of the fluctuations, and also a comparison of them with meteorological measurements.

The propagation of radio waves in a real medium is usually accompanied by amplitude and phase fluctuations. These fluctuations are brought about by changes in the dielectric constant which are connected with the spatial inhomogeneity of the medium and with the instability of its properties in time. Experimental studies of the fluctuations of radio signals can be used for the investigation of physical processes taking place in the troposphere, processes which determine the characteristics of the propagation of radio waves in the troposphere.

In researches published earlier, chief attention has been paid to amplitude fluctuations. Phase fluctuations have been studied in less detail--chiefly over dry land and within the limits of the range of direct visibility [1-3]. A small number of researches has been devoted to measurements over the sea, carried out under specific conditions [4,5]. At the same time, investigations of fluctuations under sea conditions are of especial interest, inasmuch as the complexity and manifold character of the relief of the earth usually makes much more difficult the analysis of fundamental regularities over dry land.

1 1. In the present work, fundamental results of the experimental measurement of fluctuations of phase fronts in the propagation of vertically polarized radio waves of centimeter length ($f \simeq$ 3000 Mc) over the sea are reported.* Experiments were carried

*A brief account of certain results of this research is given in Ref. 6.

out under different meteorological conditions in the summer-autumn (June-September) and autumn-winter (October-December) periods, for purely sea runs of 33 km length. In the investigations, a differential method was employed which made it possible, by means of measurement of pulsations of the phase differences of the emf which were found in spaced antennas, to determine the intensity and degree of decorrelation of the phase fluctuations at different points of the wave front. To eliminate the errors associated with the change in the geometric position of the transmitter-receiver points and the instability of the frequency, experiments were carried out on a fixed run with fixed transmitting and receiving antennas, in which the receiving antennas were located along a line perpendicular to the direction of propagation. To lessen the effect of coastal refraction, six receiving systems were employed, laid out along the shore line at 15-20 meters from the shore at fixed distances of 2, 5, 10, 30 and 100 meters relative to the first (reference) antenna. The elevation of the receiving antennas (about 4 m above sea level) was also kept fixed during the time of the experiment.

For the study of the altitude dependence, lobe switching of the transmitter antenna located at different altitudes above sea level (35, 18 and 9 m) was carried out which, for standard refraction, corresponded to the transition from "illuminated" zone to the region of "half-shadow" and "shadow."

Measurement of the fluctuations of phase differences was always carried out relative to the first receiver of the system, and simultaneously for two different distances between the receiving antennas, so that for each measurement made, there was a possibility of duplicating one of the previous distances. Such a method made it possible to take into account the effect of the non-stationary characther of the processes under consideration on the results of measurements, without having recourse to the crude method of simultaneous measurement at all altitudes and distances between antennas. The duration of each such measurement amounted to 5-10 minutes, so that measurements at all fixed distances between receiving antennas was carried out in 30 minutes for each of three altitudes of the transmitter.

With the aim of making clear the dependence of the phase fluctuations on the altitude, experiments were also carried out with a rapid change in the altitude of the transmitter antenna for different fixed distances between receivers.

During the experiments, both visual indications were employed and also automatic recording of data on photoplates and film. The error of measurement of fluctuations of phase differences amounted to no more than plus or minus one degree in the measurement of the amplitudes of the received signals up to 60 db and of frequency instability, $\sim 10^{-3}$. Long time instability of the apparatus was taken into account by means of phase calibration and regular checking of the amplitude-phase and frequency-phase characteristics of the system.

The apparatus that was employed made it possible to reproduce the spectrum of the phase fluctuations in the frequency range from 0.01 cycle to 100 cycles. Moreover, for analysis of the effect of the degree of reproduction of the spectrum on the measured characteristics the possibility was provided of filtration and separate determination of the low frequency (less that 0.3 cycle) and high frequency (greater than 0.3 cycle) spectral components, called below "slow" and "fast" fluctuations.

The measurements carried out show that with rare exception the deviations of the phase differences ϕ^0 relative to the mean value for all altitudes of the transmitter h_1 and distances between antennas d are distributed according to the normal law. Typical integrated distributions of the fluctuations of the phase differences (in magnitude) are shown in Fig. 1 for one of the experiments of the summer period in a scale which linearizes the normal distribution.*

*In Fig. 1, W = probability of the magnitude of the fluctuations exceeding ϕ^0, where ϕ^0 is the deviation of the phase difference from the mean value.

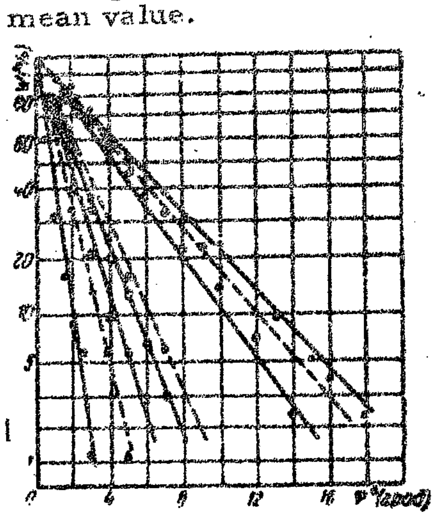

Fig. 1. Integral distribution of phase difference fluctuations (in magnitude) for different distances between receiving antennas d (h_1 = 35 m, h_2 = 34 m, r = 33 km, λ = 10 cm); experimental data: • -- d = 2 m, △▲ --d = 5 m, □■ -- d = 10 km, ○● --d = 30 m, X --d = 100 m.

As follows from the drawing, the experimental data in this scale virtually coincide with straight lines corresponding to the normal distribution with different dispersions. The data in the drawing refer to measurements with reproduction of the total spectrum of the fluctuations ("complex" fluctuations); however, similar results hold for the separate presentations "slow" and "fast" fluctuations.

It should be noted that there was naturally some lack of stationarity in the observed random fluctuations for measurements in the 5-10 minute intervals (especially in "slow" fluctuations at maximum altitudes and distances between antennas). The spectra of the fluctuations, which were concentrated in the region of very low frequencies, in contrast to the stationary random processes, were characterized by a lack of uniformity and constancy in time of the spectral density. In a sample of the present spectra for "slow" fluctuations is shown in Fig. 2, where the dependence of the ratio $\Phi^2/\overline{\Delta\phi^2}$ on the frequency F is plotted (Φ^2 is the "power" of the fluctuations in frequency band of 0.005 cycle, $\overline{\Delta\phi^2}$ is the mean square value of the total "power" of the fluctuations).

Different cases of phase fluctuations of signals were observed both as to intensity and degree of decorrelation in space. However, analysis of the measurements shown show the presence of definite tendencies in the characteristics of the fluctuations: for example, the dependence of the intensity of the fluctuations of the phase difference on the

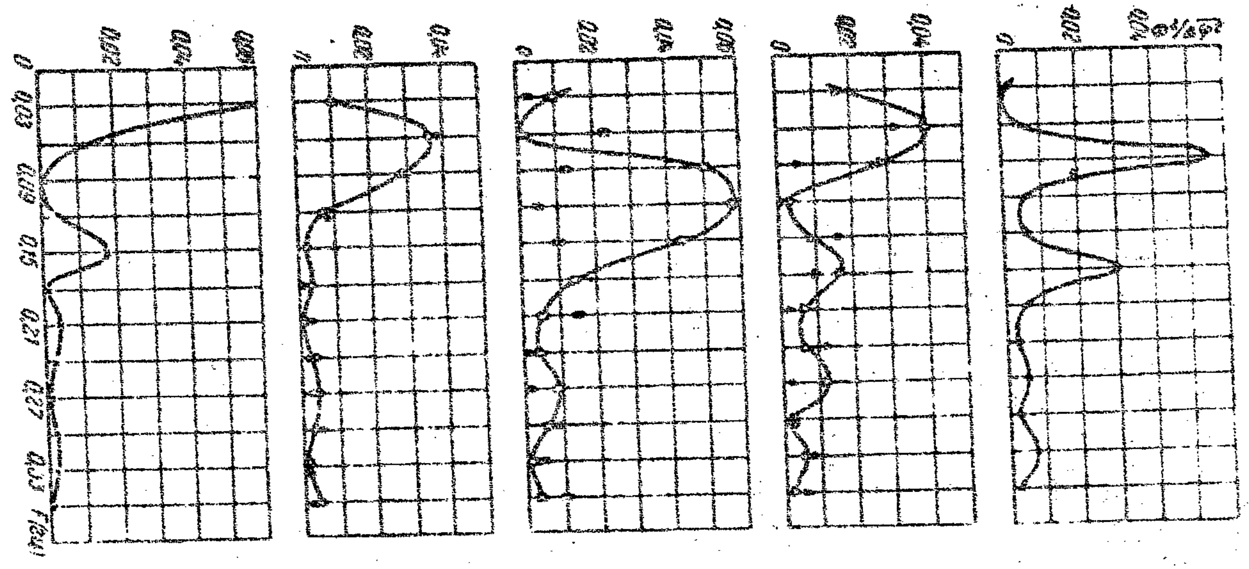

Fig. 2. Normalized energy spectra of ''slow'' fluctuations of phase differences at different distances between receiving antennas d(h_1 = 9m, h_2 = 4m, r = 33 km, λ = 10cm); experimental data: • --d = 2m, △ ▲ --d = 5m, □ ■ --d = 10m, ○ ● --d = 30m, X --d = 100m.

distance between the receiving antenna kept qualitatively virtually identical in all experiments.* A much greater variety is observed

*This characteristic of the fluctuations $\left(\overline{\Delta\phi^2}\right)^{1/2} = f(d)$ for concentration is called correlation in what follows, inasmuch as it determines the function of the spatial correlation of the fluctuations.

in the dependence of the intensity of the fluctuations on altitude of transmitter antennas over the surface of the section. For this reason, the altitude dependence can be set down on the basis of a classification of experiments. Depending on this characteristic, and also on the degree of non-stationarity, all measurements can be divided into four basic groups: a, b, c, d.

a) Experiments with quasi-stationary standard characteristics of phase fluctuations. A number of summer-autumn and the majority of autumn-winter experiments can be assigned to this type, which is characterized by a monotonic decrease in the intensity of the fluctuations with increase of height (on the average, inversely proportional to the height). It can be shown [7] that one can represent the similar height dependence for the zone of direct visibility in the form

$$\left(\overline{\Delta\varphi^2}\right)^{1/2} \sim 1/h_1^a \qquad (a \leqslant 1). \tag{1}$$

This dependence should hold for propagation over a particular surface in a locally isotropic medium (troposphere).

The relative stationarity of the intensity of fluctuations and their temporal characteristics (for example, spectral characteristics) is distinctive of these measurements. Typical for this case is the dependence obtained in one of the experiments of the summer period and shown in Fig. 3. At all altitudes, increase in the distance between the receiving antennas is initially accompanied by a proportional increase in the fluctuations (effective values); however, thereafter there is a great slowing down in the increase of the fluctuations. At large distances (d $>$10-30m), this dependence tends toward "saturation" (Fig. 3a). The absence of further increase testifies to the significant decorrelation of the fluctuations of the phase of the signals obtained in the various antennas. In this case,

$$\overline{\Delta\varphi^2} = \overline{(\varphi_1 - \varphi_l)^2} = 2\overline{\varphi_1^2}\,(1 - R_{1l}) \leqslant 2\,\overline{\varphi_1^2} \tag{2}$$

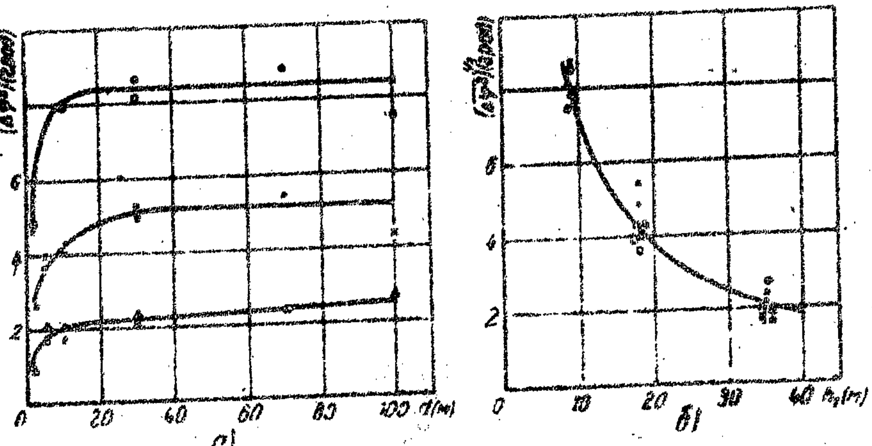

Fig. 3. Dependence of the effective values of the fluctuations of phase differences $(\overline{\Delta\phi^2})^{1/2}$ on the distance between the receiving antennas d and the altitude of the transmitter h_1 ($r = 33$ km, $h_2 = 4$m, $\lambda = 10$ cm); experimental data: a) ○ $-h_1 = 9$m, X $-h_1 = 18$m, △ $-h_1 = 35$m;
b) ○ -d = 100m, X -d = 30m, △ -d = 10m, □ -d = 5m.

Under the assumption of statistical isotropy of the medium, when at different points in space equally distant from the source (for identical altitude at these points above the surface)

$$\overline{\varphi_1^2} \simeq \overline{\varphi_i^2}. \tag{3}$$

Here ϕ_1, ϕ_i are the fluctuations of the absolute phases of the signals in the first and ith antenna, R_{1i} is the correlation coefficient of the phase fluctuations in these antennas. Thus, this dependence, normalized to its value at maximum distance between antennas d_{max}, completely determines the correlation coefficient of the fluctuations:

$$\frac{\overline{\Delta\varphi^2}}{\overline{\Delta\varphi^2_{\text{макс}}}} = \frac{\overline{(\varphi_1 - \varphi_i)^2}}{\overline{(\varphi_1 - \varphi_n)^2}} = \frac{1 - R_{1i}}{1 - R_{1n_{\text{мин}}}} \sim 1 - R_{1i}, \tag{4}$$

where the nth antenna corresponds to the distance d_{max}.

The data of Fig. 3 refer to measurements with the reproduction of the complete spectrum of the frequencies of the fluctuation ("complex" fluctuations). Similar results also take place for the separate of "slow" and "fast" fluctuations. Results of such an experiment are shown in Fig. 4. In this case, just as in the previous, a comparatively small scatter of data ($\sim$ 5-10 per cent) is observed for all repeated measurements. It is characteristic that in such measurements, the decorrelation of "slow" fluctuations always sets in much later than the "fast". *

*We note that in the "complex" measurements there is an intermediate region of correlation dependence with a much faster growth than for the "slow" fluctuations, but much slower than in the case of "fast" fluctuations.

This shows that the fluctuations in the region of very high frequencies are essentially determined by fine scale inhomogeneities of the troposphere, the spatial correlation between which disappears more rapidly than in the case of "slow" fluctuations, which are evidently brought about by much larger formations. As a consequence, the effect of the high frequency part of the spectrum of phase fluctuations is more important at small distances between the antennas (d 2-10m), where the basic "energy" of the fluctuations is determined by these frequencies. For large distances between the antennas, the "slow" fluctuations generally dominate. This conclusion is supported also by direct determination of the fluctuation spectra for different d in the case of complex measurements, and gives evidence on the deformation of the fluctuation spectrum of the phase differences under change of distance between the receiving antennas. Change of the spectral component also takes place with change of altitude of the transmitter h_1 (see, for example, the relation of the "fast" and "slow" fluctuations in Fig. 4), which is explained by the steeper increase of the "fast" fluctuations in approaching the surface of measurement in comparison with the "slow" **

**This agrees with theoretical calculations carried out for a plane surface[7].

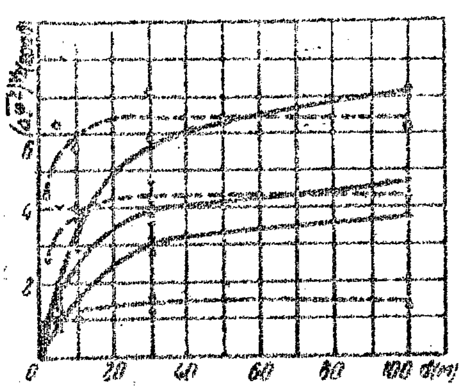

Fig. 4. Dependence of the effective values of the "slow" and "fast" fluctuations on the distance d and the altitude h_1 ($r = 33km$, $h_2 = 4m$, $\lambda = 10cm$); experimental data: a) "slow" fluctuations (· --h_1 = 9m, ✕ --h_1 = 18m, ▲ --h_1 = 35m); b) "fast" fluctuations (O --h_1 = 9m, V --h_1 = 18m, Δ --h_1 = 35m).

The "present" spectra (see, for example, Fig. 2) computed from the experimental data and the autocorrelation functions

show that the decrease in the distance d and altitude above ground leads to a relative broadening of the spectra of the fluctuations of the phase differences. In all cases there is a tendency toward a sharp decrease in the spectral density with increase in the fluctuation frequency. Between 90-95 per cent of the "energy" of the fluctuations is usually distributed in the band of frequencies to 10 cycles.

b) Non-stationary experiments with standard characteristics of phase fluctuations. These measurments differ in the sharp non-stationary character of the fluctuations. An essential change is noted in the qualitative character and intensity of the phase pulsations for repeated measurements in adjacent ten-minute intervals and even during the course of a single measurement. However, the essential dependencies and characteristics of the phase fluctuations are qualitatively the same as in the first case.

It should be emphasized that the method applied in experiments of this type makes it possible to determine in first approximation the average value of the fluctuation characteristics, inasmuch as exact data under conditions of sudden non-stationarity can be obtained only for simultaneous parallel measurements at all altitudes and distances between antennas. Such measurements were encountered only during the summer-autumn period; we note that measurements of types a) and b) comprise about 70 percent of the summer and more than 90 per cent of the winter experiments.

c) Anomalous type of characteristics of phase difference fluctuations. Measurements of this group: were characterized by a type of altitude dependence differing sharply from the standard, in which the intensity of the fluctuations increased monotonically with increase in altitude or passed through a maximum. Such results were observed both in simultaneous ("complex"), and also in separate measurements of "slow" and "fast" fluctuations; however, the anomaly of the altitude characteristics was more pronounced in the "slow" fluctuations. The characteristics of the "fast" fluctuations were distinguished as a rule by a very high stability. Data of one of such experiments in the measurement of "slow" fluctuations are given in Fig. 5. The anomaly of the dependence of the fluctuations on altitude in this experiment is most strongly expressed at maximum distances d (see Fig. 5b), in which the principal part of the "energy" of the fluctuations is brought about by large inhomogeneities; with decrease in d, the characteristic of the altitude dependence approaches the standard.

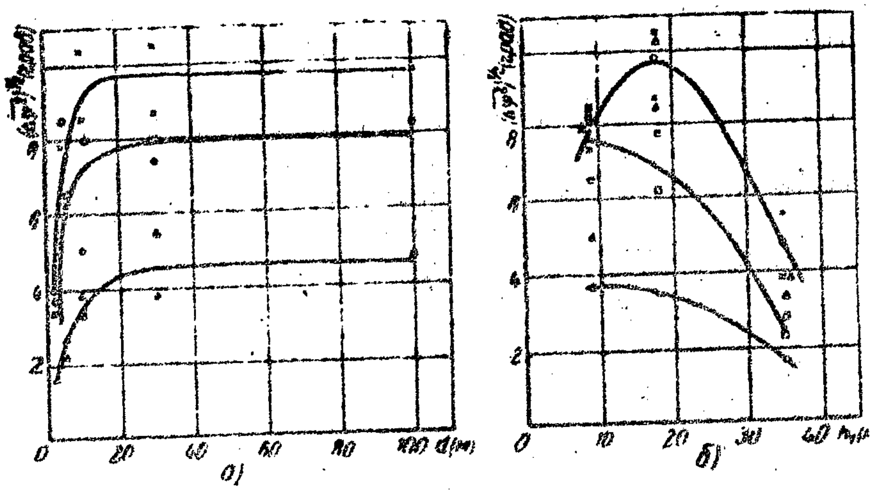

Fig. 5. Characteristics of "slow" phase fluctuations for anomalous altitude dependence ($r = 33$ km, $h_2 = 4$ m, $\lambda = 10$ cm); experimental data: a) O $-h_1 = 9$ m, X $-h_1 = 18$ m, △ $-h_1 = 35$ m, b) O $-d = 100$ m, X $-d = 30$ m, △ $-d = 10$ m, □ $-d = 5$ m, • $-d = 2$ m.

It is also characteristic that in these measurements the altitude dependence for "fast" fluctuations was close to standard. Experiments with a sharply delineated altitude dependence were encountered rather rarely (less than 10 per cent of the summer and autumn measurements). Significantly more (about 20 per cent of the experiments) were observed in the case with weakly expressed or completely absent altitude dependence which are the transitional forms from measurements of the previous two forms for experiments of type c). Results of one of these experiments for "complex" measurements of fluctuations are shown in Fig. 6. It should be noted that the disruption of the normal altitude dependence in certain cases comes about at the highest values of d: a number of cases were noted with anomalous altitude dependence for large d and normal dependence for minimum d. All these factors are evidently connected with the fact that the vertical anisotropy of the lower levels of the troposphere, which can bring about the observed changes in the dependence of the intensity of the phase fluctuations on the altitude is fundamentally determined by the large scale inhomogeneities. In measurements of this type, a sharply expressed anomaly is usually accompanied by a large non-stationarity of the fluctuatinns. Measurements of an intermediate type are encountered both of the non-stationary and stationary types.

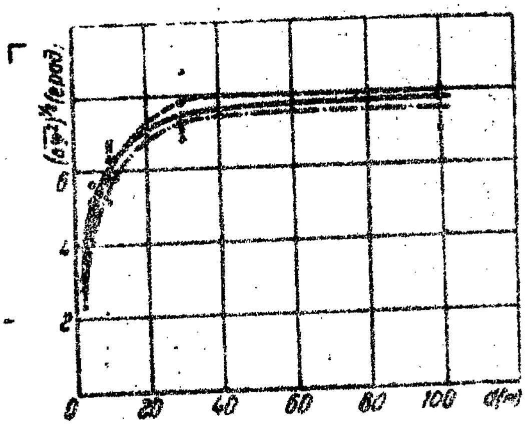

Fig. 6. Characteristics of "complex" fluctuations for weakly expressed altitude dependence; experimental data:
O $-h_1 = 9$ m, X $-h_1 = 18$ m,
Δ $-h_1 = 35$ m

~~d) Fluctuations~~

d) Fluctuation "bursts."
In some experiments of the summer period there was noted a short time, extremely sharp increase in the intensity of the phase difference fluctuations, the so-called "fluctuation bursts." Large and non-stationary fluctuations usually precede this phenomenon. For several minutes the fluctuations increase rapidly ($\Delta \phi \sim 2\pi$), and are accompanied by deep-seated fadings of amplitude. The duration of such a state as a rule does not exceed several tens of minutes, after which the ordinary picture is established. A satisfactory explanation of this phenomenon cannot be given within the framework of the theory developed for an unlimited inhomogeneous medium [8-13], since in this case one must assume (if account is not taken of the effect of the surface of separation) an increase in the fluctuations of the dielectric constant of the medium over the path by hundreds and thousands of times in comparison with the ordinary conditions which is excluded in practice. Account of the surface of separation [7] in propagation in an inhomogeneous medium leads to the conclusion that the course of such an anomalous increase in the fluctuations can for the ordinary stability of the medium (fluctuations of the dielectric constant) be the change in the mean refraction over the trajectory, which leads to an interference minimum at the point of the receiver.

2. In measurements of all types considered, along with the change in the qualitative character of the fluctuation dependence from experiment to experiment, a variation was noted in the value of the intensity of the fluctuations, in which the maximum change (not taking into account the phenomenon of "bursts") were those involving fluctuations of the phase difference for minimum distance between the receiving antennas. For illustration, the changes of the intensity of "slow" fluctuations are plotted in Fig. 7 for the summer-autumn period at distances between the receiving antennas of d = 2 and d = 100 m, and altitude of the transmitter 9m.

The increased variability of the intensity of the fluctuation observed in experiments over the surface of separation for small distances between the receiving antennas should evidently also take

place in measurements in an unbounded inhomogeneous medium, inasmuch as in such a case the fluctuations are determined with a more complicated dependence on the properties of the inhomogeneous medium, i.e., on the correlation function describing the pulsations of the coefficient of refraction in it), than for the case of large d. For small d, the intensity of the phase difference fluctuations is determined both by the fluctuations of the absolute phase of the signals and by the degree of decorrelation between them [$R_{1i} \gg 0$, see Eq. (2)] while for large d the correlation of these fluctuations is insignificant ($R_{1i} \to 0$). As shown in Refs. 8, 10, the intensity of the fluctuations of the absolute phases is determined only by the scale of the correlation function ℓ* and $\overline{n^2}$ (n = index of refraction), i.e., they are practically independent of the exact form of this function.

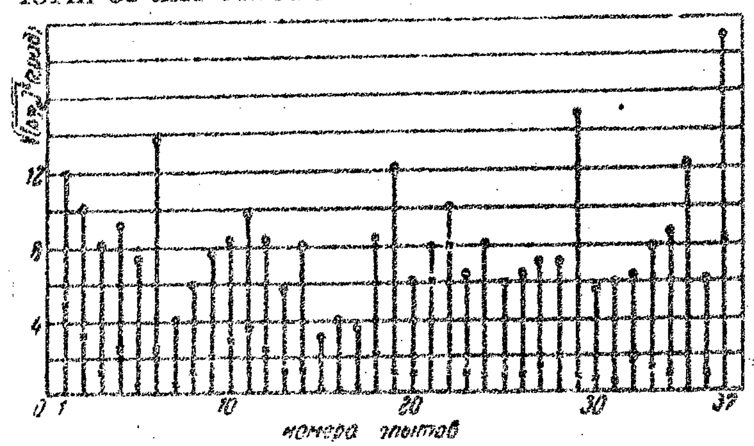

Fig. 7. Change in the intensity of "slow" fluctuations in experiments of the summer-autumn period (r = 33 km, h_1 = 9 m, h_2 = 4 m, λ = 10 cm); experimental data: X -d = 2 m, O -d = 100 m.

*That is, the distance at which a significant falling off of the correlation of the pulsations of the inhomogeneities of the medium takes place.

It is appropriate to compare the change of the intensity of the phase fluctuations noted in the experiments with the state of the medium, which is directly determined by the observed fluctuations. Some results of such a comparison of radio measurements for d = 10 m and h_1 = 9 m with the mean meteorological conditions over the path are given in Figs. 8-10. As follows from an analysis of data obtained for different distances and altitude above ground of the antenna system, it has not proved possible to establish a direct functional relation between the values of fluctuations and the average meteorological data. However, it can be remarked that an increase in the wind velocity, independent of its direction, and an increase in the waves of the sea (see Figs. 8, 9) were usually accompanied by a decrease in the intensity of the fluctuations. Decrease of the fluctuations was also noted, as a rule, upon increase in radio refraction

up to $a_{eq} \rightarrow \infty$* (see Fig. 10) and for hazy and rainy weather.

*The equivalent radius of the earth a_{eq} for an estimate of the degree of refraction was determined according to the magnitude of the mean field on the basis of the diffraction formulas of V. A. Fock.

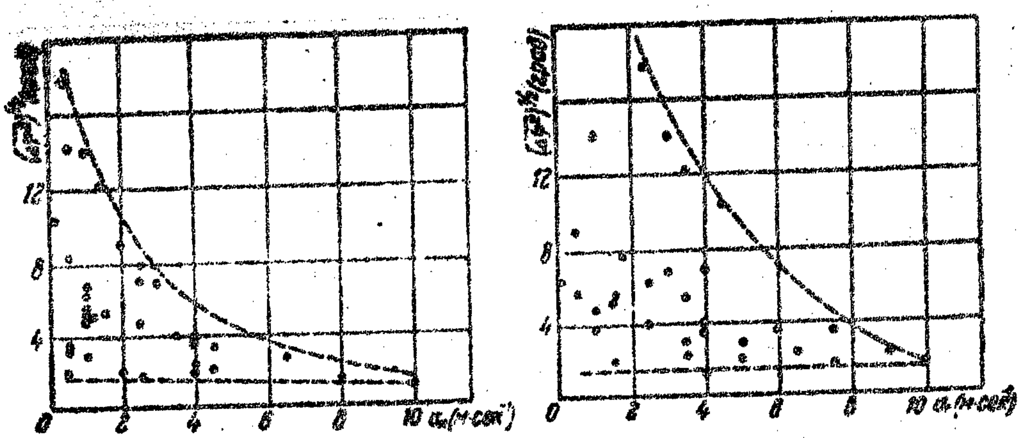

Fig. 8. Intensity of the phase difference fluctuations in wind of different force (in the measurements of the summer-autumn; r = 33km, h_1 = 9m, h_2 = 4m, d = 10m, λ = 10cm); v_n = wind component transverse to the path, v_t = wind component along the path.

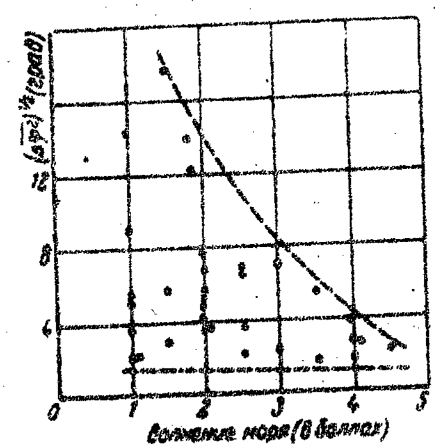

Fig. 9. Intensity of the phase difference fluctuations for different sea waves (according to measurements of the summer-autumn period; r = 33 km, h_1 = 9m, h_2 = 4m, d = 10m, λ = 10cm).

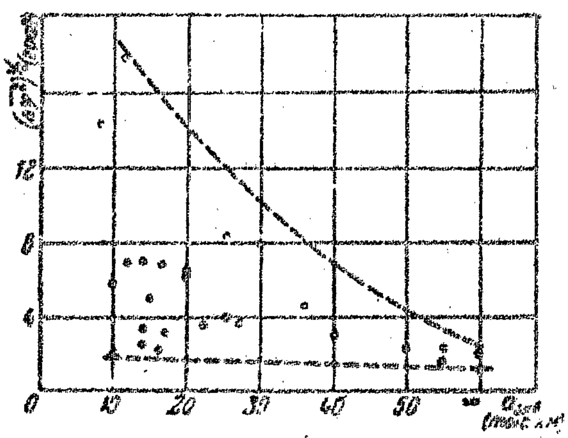

Fig. 10. Phase fluctuations for different degree of radio refraction (according to measurements of the summer-autumn period; d = 10m, h_1 = 9m, h_2 = 4m, r = 33km, λ = 10cm).

The dependence on the sea state that has been pointed out and also the fact that the maximum fluctuations were observed for windless, sunny weather and in the absence of waves, show that in the measurements that have been carried out the state of the sea surface did not have an important effect on phase fluctuations.

*The qualitative coincidence of the results obtained in the measurements over dry land also shows this result (see, for example, Ref. 2).

In conclusion, we note that the measurements that have been carried out make it possible to estimate the radii of the spatial correlation of phase fluctuations (the "scale" of the inhomogeneities) by means of a comparison of experimental data with known theoretical investigations of the propagation of radio waves in a locally isotropic medium [8-13]. One of the typical examples of such a comparison is shown in Fig. 11, where the computed dependence of the mean square of the phase difference fluctuations $\overline{\Delta\varphi_d^2}$ on the distance between receiving antennas d for different values of the "scale" of tropospheric inhomogeneities ℓ are shown by the solid line. The values of $\overline{\Delta\varphi_d^2}$ are normalized to the maximum values $(\overline{\Delta\varphi_d^2})_{max}$, corresponding to d_{max}:

$$\frac{\overline{\Delta\varphi_d^2}}{(\overline{\Delta\varphi^2_{d_{\text{макс}}}})} = \frac{1-(\sqrt{\pi}\, l/2d)\,\mathrm{erf}\,(d/l)}{1-(\sqrt{\pi}\, l/2d_{\text{макс}})\,\mathrm{erf}\,(d_{\text{макс}}/l)}. \quad (5)$$

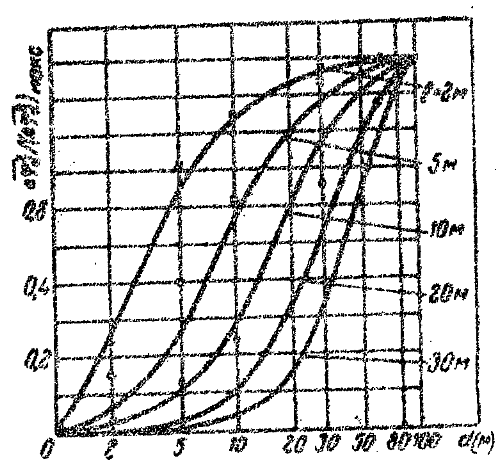

Fig. 11. Dependence of the normalized mean square values of the phase difference fluctuations on the distance d between the receiving antennas (h_1 = 18m, h_2 = 4m, r = 33km, λ = 10cm); experimental data: O - "complex" measurements, × - "fast" fluctuations, ● - "slow" fluctuations.

We note that Eq. (5) is valid in the case of an unbounded space, and also, as shown in Ref. 14, for propagation over a surface of separation* in an inhomogeneous medium described by a spatial correlation function of fluctuations of the index of refraction n of the form:

where n_1, n_2 are the fluctuations of the coefficient of refraction at different points of space, ρ is the distance between these points

ℓ is the parameter of the correlation function ("scale" of the inhomogeneities). However, as shown in Ref. 10, similar results are obtained for different forms of the correlation function in which the correlation of the pulsations of the index of refraction fall off monotonically with increase in d.

As follows from the given data, the value of ℓ for "slow" fluctuations is of the order of 10-30m, for "fast,"--1-3m. In the case in which the entire fluctuation spectrum is reproduced, an intermediate form of the dependence of the fluctuations on the distance is observed, and the parameter ℓ amounts to 3-10m, increasing somewhat with the altitude of the transmitting antenna.

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* When the distance d between the receiving antennas is measured along the line parallel to the surface of separation (perpendicular to the direction of propagation), and at high altitudes either of both ends of the path or of one of them ($h_1, h_2 \gg \ell$ or $h_1 \gg \ell$, $h_2 \gg \ell$).

Thus the narrowing of the fluctuation spectrum which takes place in inertial measurements without essentially disrupting the altitude characteristics in most cases always brings about an appreciable lowering of the rate of increase of the intensity fluctuations upon increase in the distance d. In the theoretical interpretation of the experiment, this circumstance reduces to the necessity of increasing the parameter ℓ. Moreover, separate measurement of the fluctuations in several frequency intervals of the spectrum leads to a number of vlaues of the "scale" of irregularities which bring about these fluctuations.

The authors express their gratitude to V. F. Shul'ge, O. M. Lebedev and B. F. Veber who took part in carrying out the measurements.

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INTERACTION OF ELECTROMAGNETIC WAVES IN PLASMA. II

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(Submitted to editors 16 July 1959)

(Abstract) On the basis of relations obtained in Ref. 1 between the coefficients of the asymptotic solution on both sides of regions of interaction, concrete cases are considered of the interaction of normal waves in a weakly inhomogeneous magnetoactive plasma for $\omega_H/\omega < 1$ and for $\omega_H/\omega > 1$ (ω = frequency of wave, ω_H = gyrofrequency). Explicit expressions are found for the characteristic parameters of the interaction, which are valid at small angles between the constant magnetic field and the direction of propagation of the wave.

1. PRELIMINARY REMARKS

In Ref. 1, cited below as I, the following relations were obtained between the coefficients of the asymptotic solution connecting the amplitudes of normal waves on both sides of regions of interaction in a plasma:*

$$\left.\begin{aligned} c'_{II} &= \sqrt{1-e^{-2\delta_{01}}}\, c_{II} - e^{-\delta_{01}} c_{III},\\ c'_{III} &= e^{-\delta_{01}} c_{II} + \sqrt{1-e^{-2\delta_{01}}}\, c_{III}; \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} d'_{II} &= \sqrt{1-e^{-2\delta_{01}}}\, d_{II} - e^{-\delta_{01}} d_{III},\\ d'_{III} &= e^{-\delta_{01}} d_{II} + \sqrt{1-e^{-2\delta_{01}}}\, d_{III}; \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} \tilde{c}'_{I} &= \sqrt{1-e^{-2\delta_{02}}}\, \tilde{c}_{I} + e^{-\delta_{02}} \tilde{c}_{III},\\ \tilde{c}'_{III} &= -e^{-\delta_{02}} \tilde{c}_{I} + \sqrt{1-e^{-2\delta_{02}}}\, \tilde{c}_{III}; \end{aligned}\right\} \tag{1.3}$$

$$\left.\begin{aligned} \tilde{d}'_{I} &= \sqrt{1-e^{-2\delta_{02}}}\, \tilde{d}_{I} + e^{-\delta_{02}} \tilde{d}_{III},\\ \tilde{d}'_{III} &= -e^{-\delta_{02}} \tilde{d}_{I} + \sqrt{1-e^{-2\delta_{02}}}\, \tilde{d}_{III}. \end{aligned}\right\} \tag{1.4}$$

*In paper I, these relations were found (in quasi-hydrodynamic approximation) for electromagnetic waves propagated along grad N (N = concentration of electrons) in a weakly inhomogeneous plane-layered magnetoactive plasma.

In equations (1.1)-(1.4), the characteristic parameters of the interaction δ_{01} and δ_{02} are determined by the integral

$$\delta_{01} = -i\rho \oint \frac{n_{III} - n_{II}}{4}\, d\varepsilon; \quad \delta_{02} = -i\rho \oint \frac{n_{I} - n_{III}}{4}\, d\varepsilon, \qquad (1.5)$$

which are taken over a closed curve in the complex plane ε containing the point in which the respective indices of refraction are equal to n_{III} and n_{II}, n_I and n_{III}.

In (1.5), $\varepsilon \equiv 1 - v = 1 - \omega_0^2/\omega^2$, where ω_0 is the Langmuir frequency of the plasma, $\rho = k_0/|\text{grad}\, \varepsilon|$, where $k_0 = \omega/c_0$, c_0 is the velocity of light in a vacuum.

We recall that, in Eqs. (1.1)-(1.4), c denotes the amplitude of the normal wave traveling in the direction of increase in v (in the direction of negative ε, see Fig. 1) while d is the amplitude of the wave traveling in the opposite direction. The prime on c and d denotes that these quantities are taken behind the region of interaction--at the point with large values of the parameter v; the values of c and d without the prime refer to points before the interaction--with correspondingly smaller values of v. The sign ( ) means that the quantities c and d are associated with the region of interaction between waves I and III, while the designation of the amplitudes c and d without this sign refer to the interaction of waves II and III.

The relations (1.1)-(1.4) in principle completely solve the problem of the interaction of electromagnetic waves in plane-layered magnetoactive plasma (for propagation of waves along grad ε). Making use of these relations, we shall consider concrete cases of the interaction of "normal" waves.*

*It is necessary to emphasize that the "interaction of normal waves" considered here is not connected with a violation of the principle of superposition for electromagnetic fields in a plasma, i.e., with a nonlinear character of electromagnetic waves; "interaction" is brought about by the fact that in certain regions of inhomogeneous plasma, the decomposition of the field into normal waves is not applicable.

2. INTERACTION OF "NORMAL" WAVES FOR $u < 1$ (Fig. 1a)

It is clear from Fig. 1 that there are two regions of interaction in a magnetoacoustic plasma for $v \simeq 1$. However, for $u \equiv \omega_H^2/\omega^2 < 1$, one of the regions corresponds to $n_j^2 < 0$. In the case of a sufficiently strong magnetic field one can neglect the interaction in this region, since the waves are strongly damped for $n_j^2 < 0$. The condition [2] *

$$\sqrt{u} = \frac{\omega_H}{\omega} \gg \left(\frac{c_0}{\omega} \, |\operatorname{grad} \varepsilon|\right)^{2/3}.$$

serves as a criterion under fulfilment of which interaction in the region $n_j^2 < 0$ is not at all negligible. Actually, on satisfaction of

*In the case of weakly inhomogeneous plasma, $\left(\frac{c_0}{\omega} \, |\operatorname{grad} \varepsilon|\right)^{2/3} = \rho^{-2/3} \ll 1$.

this inequality the extraordinary wave (n_1^2) emerging from the region of interaction $n_j^2 < 0$ is strongly damped in the region $n_1^2 < 0$ (i.e., in the layer $1 > v > 1 - \sqrt{u}$), since in this case the distance between $v = 1$ and $v = 1 - \sqrt{u}$ will be much greater than the length of the extraordinary wave in the region of reflection.

It is assumed below that the condition just pointed out is satisfied, i.e., the interaction for $u < 1$ takes place only between waves II and III, which correspond to positive values of n_{II}^2 and n_{III}^2 (see Fig. 1a).

a) Let only the extraordinary wave (n_2^2) with amplitude $c_{II} = 1$ be incident on the region of interaction from the side of small values of the parameter $v = \omega_0^2/\omega^2$ (the amplitude of the plasma wave $c_{III} = 0$). We shall find the amplitude of the remaining waves close to the region of interaction.

According to the conditions $c_{II} = 1$, $c_{III} = 0$, it follows from (1.1) that for waves traveling after the region of interaction in the direction of increasing v,

$$c'_{II} = \sqrt{1 - e^{-2\delta_0}}; \quad c'_{III} = e^{-\delta_0}. \tag{2.1}$$

If reflections of wave III from the point $v = 1 + \sqrt{u}$ are not taken into account, then the amplitude of the reflected wave III is equal to zero: $d'_{III} = 0$. At the same time the incident wave c'_{II} and the wave

d'_{II} reflected from the point $v = 1$ ($\varepsilon = 0$) are connected by the equation

$$d'_{II} = c'_{II} \exp\{-2i\rho S + i\pi/2\}, \qquad (2.2)$$

where $\rho S = \rho \int_{B}^{v=1} n_{II} d\varepsilon$ is the change in phase in the wave at a distance from the point B to the point $v = 1$ (see Fig. 3a of paper I), i.e., from the region of interaction of waves II-III to the point of reflection of waves II (Fig. 1a). In correspondence with (2.2), the additional change of phase by $\pi/2$ due to reflection from the point $v = 1$ was also taken into account (see Ref. 3, par. 66). According to the conditions $d'_{III} = 0$ and (2.2), it follows from (1.2) and (2.1) that

$$\sqrt{1 - e^{-2\delta_{01}}}\, e^{-2i\rho S + i\pi/2} = \sqrt{1 - e^{-2\delta_{01}}}\, d_{II} - e^{-\delta_{01}}\, d_{III};$$

$$0 = e^{-\delta_{01}}\, d_{II} + \sqrt{1 - e^{-2\delta_{01}}}\, d_{III},$$

whence

$$d_{II} = (1 - e^{-2\delta_{01}})\, e^{-2i\rho S + i\pi/2};$$
$$d_{III} = -e^{-\delta_{01}} \sqrt{1 - e^{-2\delta_{01}}}\, e^{-2i\rho S + i\pi/2}. \qquad (2.3)$$

It is clear from the relations (2.1), (2.3) that, in the incidence on the region of interaction of the extraordinary wave $n_2^2(II)$ from the side of v in the case $u = \omega_H^2/\omega^2 < 1$, the relative intensity of waves II and III, reflected from the region of interaction, will be equal to

$$|d_{II}|^2 = (1 - e^{-2\delta_{01}})^2;$$
$$|d_{III}|^2 = e^{-2\delta_{01}}\, (1 - e^{-2\delta_{01}}), \qquad (2.4)$$

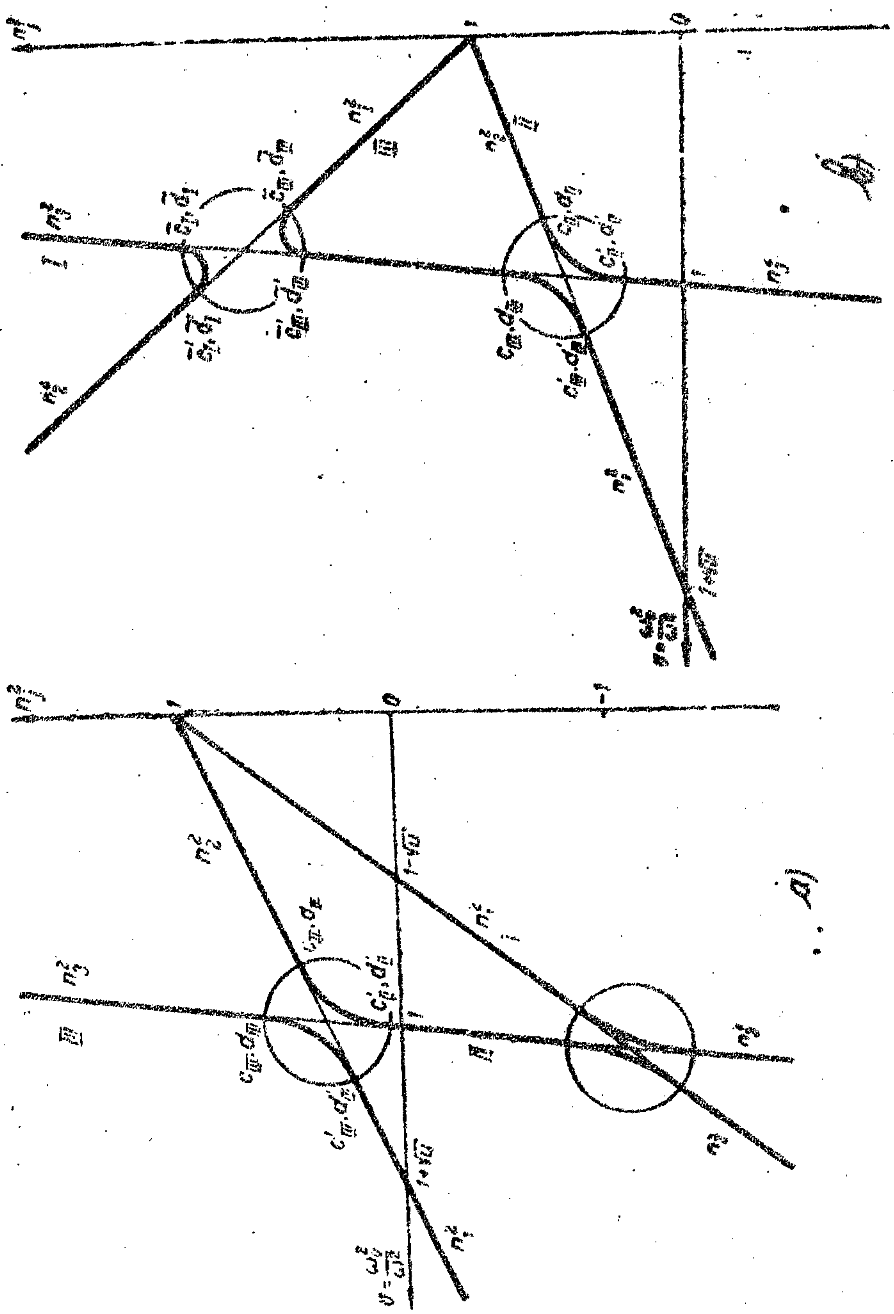

Fig. 1. Dispersion curves in the case of the propagation of electromagnetic waves at a small angle α to the direction of the magnetic field $\underline{H}_0$ (the regions of interaction of the normal waves are denoted by the circles):

a) $u = \omega_H^2/\omega^2$ [illegible] 1.   b) u [illegible] ω_H^2/ω^2 [illegible] 1.

and the relative intensity of the traveling wave III

$$|c'_{III}|^2 = e^{-2\delta_{01}}. \tag{2.5}$$

b) In the case in which the plasma wave n_3^2(III) is incident on the region of interaction, the problem is solved in similar fashion. In this case the relative intensity of the reflected waves II and III (see Fig. 1a)

$$\begin{aligned} |d_{II}|^2 &= e^{-2\delta_{01}}(1 - e^{-2\delta_{01}}); \\ |d_{III}|^2 &= e^{-4\delta_{01}}, \end{aligned} \tag{2.6}$$

while relative intensity of the wave III which travels through the retion of interattion,

$$|c'_{III}|^2 = 1 - e^{-2\delta_{0}}. \tag{2.7}$$

(Equations (2.6) and (2.) were obtained without consideration of the reflection of the ordinary wave n_1^2 from the point $v = 1 + \sqrt{u}$.)

c) If the ordinary wave n_1^2(III) approaches the region of interaction from the side of large v, then $c_{II} = c_{III} = 0$, and the relative intensity of the passing waves II and III

$$\begin{aligned} |d_{II}|^2 &= e^{-2\delta_{01}}; \\ |d_{III}|^2 &= 1 - e^{-2\delta_{01}}, \end{aligned} \tag{2.8}$$

while the intensity of the reflected wave n_1^2(III) is equal to zero:

$$|c_{III}'|^2 = 0. \tag{2.9}$$

The absence of a reflected ordinary wave in the incidence of it from the side of large v on the region of interaction was noted in Ref. 4, where some expressions were obtained (without account of thermal motion in the plasma) for the coefficients of reflection and transmission of electromagnetic waves and which have been set forth in this section.

3. INTERACTION OF NORMAL WAVES FOR u > 1 (Fig. 1b)

In the case in which u > 1, both regions of interaction correspond to the values $n_1^2 > 0$; therefore, in consideration of the interaction in a strong magnetic field account of both regions is necessary. For consideration of concrete cases of the interaction of normal waves for u > 1, one must take into consideration the fact that after the region of interaction (in the region B, see Fig. 3b of paper I) the amplitudes of the incident and reflected waves n_3^2(II) are connected by the relation (2.2). At the same time, the amplitudes of the wave n_1^2(III) at the points $\tilde{B}$ and A are connected by the relation

$$c_{III} = \tilde{c}_{III}' \exp\{-i\rho\tilde{S} + \tilde{R}\}; \quad d_{III} = \tilde{d}_{III}' \exp\{i\rho\tilde{S} + \tilde{R}\}. \tag{3.1}$$

Here

$$\rho\tilde{S} = \rho\int_{\tilde{B}}^{A} n_{III}\, dz$$

that takes into account the phase shift at the points A and $\tilde{B}$ (see Fig. 3b of paper I), i.e., in the transition of the wave III from the region of interaction of waves I and III to the region of interaction of waves II and III (Fig. 1b), while

$$\tilde{R} = \int_{\tilde{B}}^{A} R'(n_{III})\, dz \tag{3.2}$$

takes into account the change of the amplitude of the wave in the transition from $\tilde{B}$ to A (corresponding to the approximation of

geometric optics in this region; see Eq. (2.2) of paper I).

a) Let the extraordinary $n_1^2(\mathrm{III})$ with amplitude $\tilde{c}_{III} = 1$ be incident on the region of interaction from the side of small v, while the amplitude $\tilde{c}_I$ of the plasma wave $n_3^2(\mathrm{I})$, the amplitude c_{II} of the ordinary wave $n_2^2(\mathrm{II})$, and also the amplitude d'_{III} of the extraordinary wave $n_1^2(\mathrm{III})$ and the amplitude d'_I of the ordinary wave $n_2^2(\mathrm{I})$ are equal to zero (the reflection from the wave $n_1^2(\mathrm{III})$ from the point $v = 1 + \sqrt{u}$ is assumed to be absent).

We then obtain from (1.2), (1.3), (2.2) and (3.1)

$$
\begin{aligned}
c'_{II}\, e^{-2i\rho S + i\pi/2} &= \sqrt{1 - e^{-2\delta_{01}}}\; d_{II} - e^{-\delta_{01}}\, \tilde{d}'_{III}\, e^{i\rho\tilde{S} + \tilde{R}};\\
0 &= e^{-\delta_{01}}\, d_{II} + \sqrt{1 - e^{-2\delta_{01}}}\; \tilde{d}'_{III}\, e^{i\rho\tilde{S} + \tilde{R}};\\
c'_{I} &= e^{-\delta_{02}};\\
\tilde{c}'_{III} &= \sqrt{1 - e^{-2\delta_{02}}}.
\end{aligned}
\tag{3.3}
$$

At the same time, according to (1.1), (1.4) and (3.1),

$$
\begin{aligned}
c'_{II} &= - e^{-\delta_{01}}\, \tilde{c}'_{III} e^{-i\rho\tilde{S} + \tilde{R}};\\
c'_{III} &= \sqrt{1 - e^{-2\delta_{01}}}\; \tilde{c}'_{III}\, e^{-i\rho\tilde{S} + \tilde{R}};\\
0 &= \sqrt{1 - e^{-2\delta_{02}}}\, \tilde{d}_{I} + e^{-\delta_{02}}\, \tilde{d}_{III};\\
\tilde{d}'_{III} &= - e^{-\delta_{02}}\, \tilde{d}_{I} + \sqrt{1 - e^{-2\delta_{02}}}\; \tilde{d}_{III}.
\end{aligned}
\tag{3.4}
$$

Eliminating $\tilde{c}'_{III}$ and $\tilde{d}'_{III}$ from Eqs. (3.3), (3.4), we find that

As is seen from the expressions given, the amplitudes c'_{III} and d_{II} are proportional to the factor exp $\tilde{R}$, which reflects the fact of the change of the amplitude of the wave n_1^2(III) in the transition from one région of interaction to the other. However, inasmuch as the

$$\tilde{c}'_{I} = e^{-\delta_{02}};$$

$$c'_{III} = \sqrt{1-e^{-2\delta_{01}}}\,\sqrt{1-e^{-2\delta_{02}}}\; e^{-ip\tilde{S}+\tilde{R}};$$

$$\tilde{d}_{I} = -\,e^{-2\delta_{01}-2\delta_{01}}\sqrt{1-e^{-2\delta_{02}}}\; e^{-2ipS-2ip\tilde{S}+i\pi/2}; \tag{3.5}$$

$$d_{II} = -\,e^{-\delta_{01}}\sqrt{1-e^{-2\delta_{01}}}\,\sqrt{1-e^{-2\delta_{02}}}\; e^{-2ipS-2ip\tilde{S}+\tilde{R}+i\pi/2};$$

$$d_{III} = e^{-2\delta_{01}}\left(1-e^{-2\delta_{02}}\right)\, e^{-2ipS-2ip\tilde{S}+i\pi/2}.$$

radiation flux in this transition is not changed, the relative intensity of the wave n_1^2(III) passing through both regions of interaction is equal to

$$|c'_{III}|^2 e^{-2\tilde{R}} = (1-e^{-2\delta_{01}})\,(1-e^{-2\delta_{02}}), \tag{3.6}$$

while the relative intensity of the wave n_2^2(II) reflected from the region $v \simeq 1$, is

$$|d_{II}|^2 e^{-2\tilde{R}} = e^{-2\delta_{01}}\,(1-e^{-2\delta_{01}})\,(1-e^{-2\delta_{02}}). \tag{3.7}$$

Moreover, the relative intensities of the ordinary wave n_2^2(I) passing through the region of interaction, of the plasma wave n_3^2(I) reflected from the layer $v \simeq 1$, and the extraordinary wave n_1^2(III) also reflected from the layer $v \simeq 1$ are respectively equal to

$$|\tilde{c}'_{I}|^2 = e^{-2\delta_{02}};$$

$$|\tilde{d}_{I}|^2 = e^{-4\delta_{01}-2\delta_{02}}\,(1-e^{-2\delta_{02}}); \tag{3.8}$$

$$|\tilde{d}_{III}|^2 = e^{-4\delta_{01}}\,(1-e^{-2\delta_{02}})^2.$$

The other variant of the interaction can be considered in similar fashion for $u = \omega_H^2/\omega^2 > 1$.

b) If the ordinary wave $n_2^2(II)$ is incident on the layer $v \approx 1$ (from the side of small values of v), then $\tilde{c}_I = c_{III} = 0$ and $d_I' = d_{III}'$ = 0 (the reflection from the point $v = 1 + \sqrt{u}$ is not taken into account).

In this case the relative intensities are determined by the following considerations:
For the ordinary wave $n_2^2(I)$ passing through the region of interaction,

$$|\tilde{c}_I'|^2 e^{2\tilde{R}} = 0; \tag{3.9}$$

For the extraordinary wave $n_1^2(III)$ passing through the region of interaction,

$$|c_{III}'|^2 = e^{-2\delta_{01}}; \tag{3.10}$$

For the reflected plasma wave $n_3^2(I)$

$$|\tilde{d}_I|^2 e^{2\tilde{R}} = e^{-2\delta_{01}-2\delta_{02}}\,(1 - e^{-2\delta_{01}}); \tag{3.11}$$

for the reflected ordinary wave $n_2^2(II)$,

$$|d_{II}|^2 = (1 - e^{-2\delta_{01}})^2; \tag{3.12}$$

for the reflected extraordinary wave $n_1^2(III)$,

$$|\tilde{d}_{III}|^2 e^{2\tilde{R}} = e^{-2\delta_{01}}\,(1 - e^{-2\delta_{01}})\,(1 - e^{-2\delta_{02}}). \tag{3.13}$$

c) If the plasma wave $n_3^2(I)$ is incident on the layer $v \simeq 1$ (from the side $v < 1$), then $c_{II} = \tilde{c}_{III} = 0$ and $\tilde{d}_I' = d_{III}' = 0$ (reflection from the point $v = 1 + \sqrt{u}$ is not taken into account). Then the relative intensity is equal:
for the ordinary wave $n_2^2(I)$ which passes through the region of interaction

$$|\tilde{c}_I'|^2 = 1 - e^{-2\delta_{02}}; \tag{3.14}$$

for the extraordinary wave $n_1^2(III)$ passing through the region of interaction

$$|\tilde{c}_{III}'|^2 e^{-2\tilde{R}} = e^{-2\delta_{02}}\,(1 - e^{-2\delta_{01}}); \tag{3.15}$$

for the reflected plasma wave $n_3^2(\mathrm{I})$

$$|\tilde{d}_{\mathrm{I}}|^2 = e^{-4\delta_{01}-4\delta_{02}}; \tag{3.16}$$

for the reflected ordinary wave $n_2^2(\mathrm{II})$

$$|\tilde{\tilde{d}}_{\mathrm{II}}|^2\, e^{-2\tilde{R}} = e^{-2\delta_{01}-2\delta_{02}}\,(1-e^{-2\delta_{01}}); \tag{3.17}$$

for the reflected extraordinary wave $n_1^2(\mathrm{III})$

$$|\tilde{d}_{\mathrm{III}}|^2 = e^{-4\delta_{01}\;2\delta_{02}}\,(1-e^{-2\delta_{01}}). \tag{3.18}$$

d) If the ordinary wave $n_2^2(\mathrm{I})$ is incident on the layer $v \simeq 1$ (from the side $v > 1$), then $d'_{\mathrm{III}} = c'_{\mathrm{I}} = c_{\mathrm{II}} = c'_{\mathrm{III}} = 0$. In this case there is no reflected ordinary wave $n_2^2(\mathrm{I})$ and extraordinary wave $n_1^2(\mathrm{III})$:

$$|\tilde{c}'_{\mathrm{I}}|^2 = 0; \quad |c'_{\mathrm{III}}|^2 = 0, \tag{3.19}$$

and the remaining components are characterized by the following expressions:
for a plasma wave $n_3^2(\mathrm{I})$ traveling in the region $v < 1$,

$$|\tilde{d}_{\mathrm{I}}|^2 = 1 - e^{-2\delta_{02}}; \tag{3.20}$$

for the ordinary wave $n_2^2(\mathrm{II})$ passing through the region

$$|d_{\mathrm{II}}|^2\, e^{-2\tilde{R}} = 0; \tag{3.21}$$

for the extraordinary wave $n_1^2(\mathrm{III})$ passing through the region

$$|\tilde{d}_{\mathrm{III}}|^2 = e^{-2\delta_{02}}. \tag{3.22}$$

e) If the extraordinary wave $n_1^2(\mathrm{III})$ is incident on the layer $v \simeq 1$ (from the side $v > 1$), then $d'_{\mathrm{I}} = 0$ and $\tilde{c}_{\mathrm{I}} = c_{\mathrm{II}} = \tilde{c}_{\mathrm{III}} = 0$. In this case the reflected wave does not appear:

$$|c'_{\mathrm{I}}|^2 = 0; \quad |c'_{\mathrm{III}}|^2 = 0, \tag{3.23}$$

while the intensity of the transmitted wave in the region $v < 1$ is determined by the following considerations:
for the plasma wave $n_3^2(\mathrm{I})$

$$|\widetilde{d}_{\mathrm{I}}|^2 e^{2\widetilde{R}} - e^{-2\delta_{02}} (1 - e^{-2\delta_{01}}); \tag{3.24}$$

for the ordinary wave $n_2^2(\mathrm{II})$

$$|d_{\mathrm{II}}|^2 = e^{-2\delta_{01}}; \tag{3.25}$$

for the extraordinary wave $n_1^2(\mathrm{III})$

$$|\widetilde{d}_{\mathrm{III}}|^2 e^{2\widetilde{R}} = (1 - e^{-2\delta_{01}}) (1 - e^{-2\delta_{02}}). \tag{3.26}$$

We note that inasmuch as in cases d) and e) there is no extraordinary wave $n_1^2(\mathrm{III})$ reflected from the region $v \simeq 1$, account of reflection from no $n_1^2(v)$ at the point $v = 1 + \sqrt{u}$ has no effect on the results of the last two sections.

Of the variance of the interaction considered, cases b) for $u < 1$ and c) for $u > 1$ have the greatest interest for the theory of the sporadic radio radiation of the sun and planets; these determine the effectiveness of the transmission of plasma waves in electromagnetic radiation (in ordinary and extraordinary waves).

If $u < 1$, then in the incidence of a plasma wave on a region of interaction $v \simeq 1$ in the region where $v < 1$ there appears only the ordinary wave, which is propagated in the direction of small v (i.e., small values of the concentration of electrons N). The relative intensity of this wave, according to Eq. (2.6), is equal to

$$e^{-2\delta_{02}} (1 - e^{-2\delta_{01}}). \tag{3.27}$$

This expression vanishes for $2\delta_{01} = 0$ and $2\delta_{01} \to \infty$ and reaches a

maximum for $2\delta_{01} = \ln 2$ (the maximum value of (3.27) is equal to 1/4).

If $u > 1$, then in the incidence of a plasma wave on the region of interaction $v \simeq 1$ from this region in the direction $v < 1$, there appears both an ordinary and an extraordinary wave, and the relative intensities of these waves, which determine the effectiveness of the transition of the plasma wave into electromagnetic radiation, are respectively equal to

$$e^{-2\delta_{01}\ 2\delta_{02}} \left(1 - e^{-2\delta_{01}}\right); \quad e^{-4\delta_{01} - 2\delta_{02}} \left(1 - e^{-2\delta_{02}}\right). \tag{3.28}$$

The expressions that have been given also vanish in the case $2\delta_{01,02} \to 0$ and $2\delta_{01,02} \to \infty$, and achieve a maximum when $2\delta_{01,02} \sim 1$. In the latter case, the efficiency of the transformation is of the order of unity.

The problem of the efficiency of emission of radio radiation beyond the limits of the sun's corona, as a consequence of the regular interaction of waves in a magnetoactive plasma, is treated in detail on the basis of Eqs.(3.27), (3.28) in Ref. 2 (see also Ref. 5).

We note that it is not difficult to study the problem of the possibility of the transmission of long-wave cosmic radiation through the ionosphere by means of the relations that are set down in this section, and also to study the effect of "multiplication" of signals in the ionosphere for $u > 1$.

Inasmuch as the character of the dispersion curve $n_j^2(v)$ in the case $u > 1$ is materially different from the behavior of the dispersion curve for $u < 1$, the appearance of "multiplication" of signals for $u > 1$ will possess certain peculiarities which do not take place in the case of $u < 1$ (see Fig. 1a and 1b). The effect of "multiplication" of signals for $u < 1$ consists in the appearance of reflected triplets in sounding of the ionosphere at frequencies $\omega > \omega_H$. In this case, the first pulse corresponds to the extraordinary wave, while the second and third to the ordinary [4]. In sounding the ionosphere at lower frequencies $\omega < \omega_H$, ($u > 1$), the reflected signal is a doublet. The first pulse of the doublet arises as a result of the reflection of the wave from the point $v = 1$, the second as a consequence of reflection from the point $v = 1 + \sqrt{u}$. It is not difficult to conclude from the variance of transmission of electromagnetic waves to the layer $v \simeq 1$ given above that in the composition of both reflected pulses there are both ordinary and extraordinary components even in the case when the

sounding pulse contains only one of the normal components. Moreover, each pulse of the doublet will be split as a result of the difference in time of the group retardation of the ordinary and extraordinary waves in the region $0 < v < 1$.

4. CHARACTERISTIC PARAMETERS OF THE INTERACTION δ_{01} AND δ_{02} FOR SMALL ANGLES BETWEEN THE MAGNETIC FIELD AND THE DIRECTION OF PROPAGATION OF THE WAVE

It follows from what was outlined above that the effective interaction of normal waves takes place under the condition that $2\delta_{01,02} \sim 1$. The characteristic parameter $\delta_{01,02}$ depends on the properties of the plasma in the interaction region $v' \simeq 1$, i.e., on the frequency of the normal waves (which determine the location of the interaction region in the plasma), on the gradient of the electron concentration, on the magnitude and direction of the magnetic field H_0. The conditions pointed out prevent definite limitations to these quantities. However, calculation of the parameters δ_{01} and δ_{02} given by Eqs. (1.5) in the form of contour integrals in the complex plane $\varepsilon = 1 - v$ are very difficult in the general case. Determination of $\delta_{01,02}$ is simplified if $2\delta_{01,02} \ll 1$. Corresponding expressions will be given below for $2\delta_{01,02}$ found under these conditions.

The expression for $2\delta_{01,02}$ can be found in explicit form (for small values of $2\delta_{01,02}$) in the following way.

According to (2.5) and (3.10), in the incidence of an ordinary wave (from the side $v < 1$) on the layer $v \simeq 1$, the transmission coefficient of this wave through the region of interaction (in the form of the extraordinary component) is equal to

$$|c'_{III}|^2 = e^{-2\delta_{01}} \simeq 1 - 2\delta_{01} \quad (u \lesssim 1). \tag{4.1}$$

At the same time it follows from (3.8) that if the extraordinary wave is incident on the layer $v \simeq 1$ (from the side $v < 1$), then the transmission coefficient of this wave through the interaction region (in the form of the ordinary component) is

$$|c'_{I}|^2 = e^{-2\delta_{02}} \simeq 1 - 2\delta_{02} \quad (u > 1). \tag{4.2}$$

In the transition to the latter equalities in (4.1)-(4.2) it is assumed that $2\delta_{01,02} \ll 1$.

On the other hand, in the case $2\delta_{01,02} \ll 1$, which corresponds to almost complete transmission of the ordinary and

extraordinary waves through the layer $v \cong 1$, the transmission coefficients can be computed by the perturbation method, taking as the zero approximation the corresponding ordinary or extraordinary waves in longitudinal propagation (which corresponds to the case of total transmission).

In the consideration of interaction in the region of small angles α between the magnetic field $\underset{\sim}{H}_0$ and grad ε, it is convenient to start out from the equations

$$Z_1'' + \rho^2 a_1 Z_1 = \rho^2 b_1 Z_3;$$

$$Z_2'' + \rho^2 a_2 Z_2 = \rho^2 b_2 Z_3;$$

$$\beta_T^2 Z_3'' + \rho^2 a_3 Z_3 = \rho^2 d_1 Z_1 + \rho^2 d_2 Z_2, \quad (4.3)$$

which are obtained from the set (1.9) of paper I by the substitution

$$Z_1 = E_x + iE_y; \quad Z_2 = E_x - iE_y; \quad Z_3 = E_z \quad (4.4)$$

(E_x, E_y, E_z are the components of the electric field in the plasma; the z axis coincides with the direction of propagation of the wave, i.e., with the direction grad ε). The coefficients a, b and d are expressed in terms of the coefficients of the set (1.9) of paper I by the relations $a_1 = P - Q$; $a_2 = P + Q$; $a_3 = L$;

$$b_1 = -G - iG^*; \quad b_2 = -G + iG^*;$$

$$d_1 = \frac{\varepsilon - 1}{2i}(G + iG^*); \quad d_2 = \frac{\varepsilon - 1}{2i}(-G + iG^*). \quad (4.5)$$

The meaning of the quantities P, Q, L, G and G^* is clear from the formula (1.8) in I. Here we only note that in the case of longitudinal propagation ($\alpha = 0$) $b_1 = b_2 = d_1 = d_2 = 0$, and the set (4.3) decomposes into three independent equations which describe the ordinary, extraordinary and plasma waves. The latter circumstance guarantees the application of the perturbation method to the system (4.3).

Acting by means of the method described, we can obtain from Eqs. (4.3) the result* that the transmission coefficient of the ordinary wave is equal to

$$|\tilde{c}_1|^2 \sim 1 - \frac{\pi\rho\,\omega_y^2}{(1+\omega_z)^2\, n_2(v=1)}, \quad (4.6)$$

*We omit the intermediate calculations; they are entirely analogous to the corresponding calculations in Ref. 3, par. 79 in the consideration of the effect of "multiplication" of signals in the case of propagation of radio waves in the ionosphere, close to longitudinal.

We note that Eq. (4.6) (with account of (4.7)) for the leakage coefficient of the ordinary wave coincides with Eq. (79.14) in Ref. 3.

---

where $\omega_y = |e|\, H_{0y}/mc_0\,\omega; \quad \omega_z = |e|\, H_{0z}/mc_0\,\omega;$

$$v = \omega_0^2/\omega^2 = 4\pi e^2 N/m\,\omega^2;$$

$$n_2^2 = 1 - v\,(1 + \omega_z)^{-1}; \quad n_2\,(v = 1) = \omega_z^{1/2}\,(1 + \omega_z)^{-1/2} \qquad (4.7)$$

(e and m are the charge and mass of the electron, N is the concentration of the electrons in the plasma, H_{0y} and H_{0z} are the projections of the magnetic field $\underset{\sim}{H}_0$ on the coordinate axes y and z taken in such fashion that $H_{0x} = H_{0y}$).

In the incidence of the extraordinary wave the same coefficient of transmission is equal to

$$|\tilde{c}_1'|^2 \simeq 1 - \frac{\pi p\,\omega_y^2}{(1-\omega_z)^2\; n_1\,(v = 1)}, \qquad (4.8)$$

where

$$n_1^2 = 1 - v\,(1 - \omega_z)^{-1}; \quad n_1^2\,(v = 1) = \omega_z^{1/2}\,(\omega_z - 1)^{-1/2}. \qquad (4.9)$$

Comparison of the relations (4.6)-(4.9) just obtained with the formulas (4.1)-(4.2) shows that without consideration of thermal motion in the case $2\delta_{01,02} \ll 1$, the characteristic parameters of the interaction are

$$2\delta_{01,02} \simeq \frac{\pi p\,\omega_y^2}{\omega_z^{1/2}\,(\omega_z \pm 1)^{3/2}} \qquad (4.10)$$

(the upper sign refers to δ_{01}, the lower sign to δ_{02}). Taking it into account that $\rho = \frac{\omega}{c_0}\,|\text{grad}\,\varepsilon|^{-1}$ and $2\omega_y^2/\omega_z^2 = \text{tg}^2\alpha$ (inasmuch as $H_{0y} = H_{0x}$), we represent the formula (4.10) in the following form:

$$2\delta_{01,02} \simeq \frac{\pi}{2}\,\frac{\omega}{c_0\,|\text{grad}\,\varepsilon|}\,\frac{\alpha^2}{(1 \pm \omega/\omega_H)^{3/2}}. \qquad (4.11)$$

Here it has been taken into account that in the region of applicability of Eq. (4.10) (for $\rho \gg 1$ and $2\delta_{01,02} \ll 1$), the angle $\alpha \ll 1$, and we can set $\tan\alpha \simeq \alpha$, $\cos\alpha \simeq 1$ and $\omega_z \simeq \omega_H/\omega$ in (4.11).

We emphasize that the coefficients $|c_{III}'|^2$ and $|\tilde{c}_I'|^2$ (see relations (4.6), (4.8)) are obtained here by the perturbation method under the condition that $s = \nu_{eff}/\omega \to 0$ and without account of

thermal motion, i.e., for $\beta_T^2 = \varkappa T/mc_0^2 = 0$ (ν_{eff} is the effective number of collisions in the plasma with temperature T, $\varkappa$ is Boltzmann's constant, c_0 is the speed of light). Inasmuch as the character of the propagation of electromagnetic waves for small s and β_T changes insignificantly, we can assume that the corrections to the values δ_{01} and δ_{02} thus written (4.10) will be small for sufficiently small values of s and β_T.

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THE HAMILTONIAN METHOD IN THE ELECTRODYNAMICS OF ANISOTROPIC ABSORBING MEDIA

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(abstract) Hamilton's method is considered in the research in application to the electrodynamics of point charges moving in absorbing, dissipative, anisotropic media. Differential equations are obtained for the field coordinates. As an illustration, the field of a dipole is found for an isotropic absorbing medium.

In the work of Ginzburg (1), a general method was developed for finding fields of point charges in anisotropic media that bears the conventional name of Hamilton's method. In paper (2), Hamiltoh's method was extended to the case of a gyrotropic medium.

The extension of Hamilton's method to the absorptive case presents further interest. In this paper, the case is considered of the electrodynamics of an anisotropic medium with absorption. In this case, dispersion of the medium is treated more systematically than in the researches (1,2).

1. FIELD EQUATIONS

The field equations of point charges with account of conduction currents have the form:

$$\mathrm{rot}\, H = \frac{4\pi}{c} \sum_k e_k V_k \delta(x - x_k) + \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial D}{\partial t}; \qquad (1)$$

$$\operatorname{div} \boldsymbol{D} = 4\pi \sum_k e_k \delta(x - x_k) + 4\pi\rho ; \qquad (1)$$

$$\operatorname{rot} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} ;$$

$$\operatorname{div} \boldsymbol{H} = 0 . \qquad (2)$$

In equations (1)-(2), the magnetic permeability μ is set equal to 1; the index k refers to the k-th point charge.

For simplicity, we shall assume that the principal axis of the dielectric tensor $\hat{\epsilon}_{ik}$ and the conductivity tensor $\hat{\sigma}_{ik}$ are identical. We shall consider dispersion by assuming ϵ_{ik}, σ_{ik} to be functions of the frequency. We shall write down a linear relation between the vectors D and E, j and E in the following form:

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad \boldsymbol{j} = \sigma \boldsymbol{E}. \qquad (3)$$

In the system of coordinates whose axes coincide with the principal axes of the tensors $\epsilon_{ik}, \sigma_{ik}$, equations (3) are written in the form:

$$D_\alpha = \varepsilon_\alpha E_\alpha \, . \; j_\alpha = \sigma_\alpha E_\alpha \qquad (\alpha = 1, 2, 3). \qquad (3a)$$

Here $\hat{\epsilon}_\alpha$ and $\hat{\sigma}_\alpha$ are, respectively, the principal values of the tensors $\hat{\epsilon}_{ik}$ and $\hat{\sigma}_{ik}$, which are operators acting on the components of the electric field as functions of time (which is reflected in the writing of the symbol $\wedge$). In this case, we are dealing with a field which depends on time according to the law exp ($i\omega t$), and the action of the operators reduces to a multiplication of the field by the value of the dielectric constant or the conductivity for a given value of the frequency ω, i.e., $\hat{\epsilon} \exp(i\omega t) = \epsilon(\omega) \exp(i\omega t)$.

The current density j and the charge ρ are connected by the equation of continuity

$$\operatorname{div} \boldsymbol{j} = -\partial\rho/\partial t. \qquad (4)$$

Hence

$$\rho = -\int \operatorname{div} \boldsymbol{j}\, dt + \rho_0 ; \qquad (5)$$

taking into account only the volume charge, which is connected with the current density j, we set ρ_0 in equation (5) equal to zero.

We now write the field equations in a form that is more suitable for our purposes:

$$\operatorname{rot} H = \frac{4\pi}{c} \sum_k e_k V_k \delta(x - x_k) + \frac{1}{c} \varepsilon_{\text{I}}' E ; \quad (1a)$$

$$\operatorname{div}(\varepsilon_{\text{II}}' E) = 4\pi \sum_k e_k \delta(x - x_k),$$

where the operators

$$\tilde{\varepsilon}_{\text{I}}' = \left(\frac{\partial}{\partial t} \tilde{\varepsilon} + 4\pi\sigma \right); \quad \varepsilon_{\text{II}}' = \left(\varepsilon + 4\pi\sigma \int \ldots dt \right) \quad (6)$$

do not require special explanation. The operators $\partial/\partial t$ and $\hat{\varepsilon}$ and $\hat{\sigma}$ and $\int \ldots dt$ commute.

We introduce the potentials of the field A and ϕ in the usual fashion, so that

$$E = -\frac{1}{c} \frac{\partial A}{\partial t} - \operatorname{grad} \varphi ; \quad H = \operatorname{rot} A ; \quad (7)$$

the second pair of Maxwell's equations (2) is automatically satisfied in this case.

As in (1), it is convenient here to make use of a Coulomb calibration of the type div A = 0, which, for the case of an anisotropic, absorbing medium considered here, is generalized and takes the form:

$$\sum_\alpha \varepsilon_{\text{I}\alpha}' \frac{\partial A_\alpha}{\partial x_\alpha} = 0. \quad (8)$$

This condition can also be written in the form

$$\sum_\alpha \varepsilon_{\text{II}\alpha}' \frac{\partial A_\alpha}{\partial x_\alpha} = 0 .$$

For a harmonic time dependence,

$$\sum_\alpha \varepsilon_\alpha'(\omega) \frac{\partial A_\alpha}{\partial x_\alpha} = 0,$$

where

$$\varepsilon_\alpha'(\omega) = \varepsilon_\alpha(\omega) + \frac{4\pi\sigma_\alpha(\omega)}{i\omega}$$

is the complex dielectric constant.

The second equation (1a) is written in the form

$$\sum_\alpha \varepsilon_{\text{II}\alpha}' \frac{\partial^2 \varphi}{\partial x_\alpha^2} = -4\pi \sum_k e_k \delta(x - x_k). \quad (9)$$

By decomposing the solution into a Fourier integral, it is easy to find the solution of equation (9) in the form of an integral:

$$\varphi(x,t)=\frac{1}{2\pi}\sum_{k} e_k \int\int_{-\infty}^{+\infty} \frac{e^{i\omega(t-\tau)}\, d\omega\, d\tau}{\sqrt{\varepsilon_1'(\omega)\,\varepsilon_2'(\omega)\,\varepsilon_3'(\omega)}\sqrt{\sum_{\alpha}\left[\varepsilon_\alpha'(\omega)\right]^{-1}|x_\alpha - x_\alpha(\tau,k)|^2}}. \qquad (10)$$

In the absence of dispersion and absorption, a well-known result is obtained:

$$\varphi(x,t)=\sum_{k}\frac{e_k}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\sqrt{\sum_{\alpha}\varepsilon_\alpha^{-1}|x_\alpha - x_\alpha(k,t)|^2}}. \qquad (11)$$

We now return to the first equation of (1a). Upon substitution of the potential, this takes the form:

$$\Delta A-\frac{1}{c^2}\sum_{\alpha} e_\alpha\,\varepsilon_{1\alpha}'\frac{\partial A_\alpha}{\partial t}-\frac{1}{c}\sum_{\alpha} e_\alpha\,\varepsilon_{1\alpha}'\frac{\partial\varphi}{\partial x_\alpha}-\nabla\operatorname{div}A=-\frac{4\pi}{c}\sum_{k} e_k V_k\delta(x-x_k). \qquad (12)$$

Determination of the solution of this equation is the problem of the following section.

2. HAMILTON'S METHOD FOR THE DETERMINATION OF FIELDS IN AN ABSORBING MEDIUM

As usual, we shall seek the solution of equation (12) in the form of a Fourier series, assuming the field to be periodic in a cube of edge L = 1:

$$A(r,t)=\sum_{\lambda i} A_{\lambda i}\,, \qquad (13)$$

where

$$A_{\lambda i}=\sqrt{4\pi c^2}\; a_{\lambda i}\, q_{\lambda i}(t)\, e^{ik_\lambda r}\;. \qquad (14)$$

For natural waves in anisotropic, absorbing media, it is not the electric induction that is transverse, but the vector $D_n = D - i(4\pi/\omega)\, j$, where $D_{n\alpha} = \epsilon_\alpha'(\omega) E_\alpha(\omega)$, where $\epsilon_\alpha' = \epsilon_\alpha - i(4\pi\sigma_\alpha/\omega)$ (see, for example, Ref. (3)).

The vectors $B_{\lambda i}$, corresponding to the vector D_n, are introduced by expanding the vector $B = \hat{\varepsilon}_1' A$ in a series:

$$B=\sum_{\lambda i}\varepsilon_1' A_{\lambda i}=\sum_{\lambda i} B_{\lambda i}\,. \qquad (15)$$

According to (8), div B = 0, and consequently, $(k_\lambda B_{\lambda i}) = 0$. As independent polarizations, it is necessary to take the mutually orthogonal vectors $B_{\lambda 1}$ and $B_{\lambda 2}$, i.e.,

$$(B_{\lambda 1}\, B_{\lambda 2}) \quad 0.$$

This condition is similar to the well-known condition which is satisfied by normal waves in anisotropic media and which has the form $(D_1 D_2) = 0$ for gyrotropic media. Here D_1 and D_2 are the electric induction (displacement) vectors in two normal waves, corresponding to different values of the index of refraction (see, for example, Ref. 6)).

To sum up, the choice of the vectors $a_{\lambda i}$ in direction is limited by the following conditions

$$\sum_\alpha \hat{\varepsilon}'_{i\alpha} (k_\lambda)_\alpha (A_{\lambda i})_\alpha = 0; \tag{16}$$

$$\sum \hat{\varepsilon}'_{i\alpha} (A_{\lambda 1})_\alpha \, \varepsilon'_{i\alpha} (A_{\lambda 2})_\alpha = 0.$$

Furthermore, let us introduce the condition $(B_{\lambda i}\, A_{\lambda 2}) = 0$, i.e.,

$$\sum_\alpha \hat{\varepsilon}'_{i\alpha} (A_{\lambda 1})_\alpha (A_{\lambda 2})_\alpha = 0, \tag{17}$$

which, together with the conditions (16) signifies the coplanarity of the vectors $B_{\lambda i}$, $A_{\lambda i}$, k_λ.

Obtaining of the equations for $q_{\lambda i}$ is carried out by the usual method: after substitution of the sum (13) in (12), both parts of the equation are multiplied by $\sqrt{4\pi c^2}\; a_{mn}\, e^{-ik_m r}$ and integrated over the volume of the cube. The condition (17) guarantees the separation of the equations for $q_{\lambda i}$ with different i. As a result, the following oscillatory equations are obtained for the field coordinate $q_{\lambda i}$:

$$\sum_\alpha (a_{\lambda i})^2_\alpha \hat{\varepsilon}'_{i\alpha} \ddot{q}_{\lambda i} + \nu^2_{\lambda i} q_{\lambda i} = \sqrt{4\pi} \sum_k e_k (V_k a_{\lambda i}) e^{-ik_\lambda r_k}, \tag{18}$$

where

$$\nu^2_{\lambda i} = k^2_\lambda c^2 n^{-2}_{\lambda i} = c^2 \left[k^2_\lambda a^2_{\lambda i} - (a_{\lambda i} k_\lambda)^2 \right], \tag{19}$$

or

$$\sum_\alpha (a_{\lambda i})^2_\alpha \hat{\varepsilon}_\alpha \ddot{q}_{\lambda i} + \sum_\alpha 4\pi (a_{\lambda i})^2_\alpha \hat{\sigma}_\alpha \dot{q}_{\lambda i} + \nu^2_{\lambda i} q_{\lambda i} = \sqrt{4\pi} \sum e_k (V_k a_{\lambda i}) e^{-ik_\lambda r_k}. \tag{18a}$$

The difference from the equations of electrodynamics of anisotropic media without absorption lies in the presence of the term with $\dot{q}_{\lambda i}$. The dispersion is brought about by the operator character of the coefficients of the equation (we have the operators

$\sum_{\alpha} (a_{\lambda i})_{\alpha}^{2} \hat{\varepsilon}_{\alpha}$ and $\sum_{\alpha} (a_{\lambda i})_{\alpha}^{2} \hat{\sigma}_{\alpha}$ in place of constant coefficients.

The equation is simplified in the case of a dispersive, isotropic medium:

$$\hat{\varepsilon}\ddot{q}_{\lambda i} + 4\pi\hat{\sigma}\dot{q}_{\lambda i} + \frac{\nu_{\lambda i}^{2}}{a_{\lambda i}^{2}} q_{\lambda i} = \sqrt{4\pi}\,\frac{1}{a_{\lambda i}^{2}} \sum_{k} e_{k} (V_{k} a_{\lambda i}) e^{-ik_{\lambda} r_{k}}. \tag{20}$$

Inasmuch as the natural vibrations are damped, we shall be interested in forced solutions of equation (18). In the case of a harmonic right-hand side exp $(i\,\nu_0 t)$, the forced solutions do not differ from the solutions of the corresponding equations with coefficients $\sum_{\alpha} (a_{\lambda i})_{\alpha}^{2} \varepsilon_{\alpha}(\nu_0)$ and $\sum_{\alpha} (a_{\lambda i})_{\alpha}^{2} \sigma_{\alpha}(\nu_0)$.
We note that we can find a partial solution of the equation

$$\hat{\varepsilon}\ddot{x} + \hat{\sigma}\dot{x} + \nu^2 x = f(t)$$

by using the expansion of the solution in a Fourier integral:

$$x(t) = \int_{-\infty}^{+\infty} \frac{f_{\omega} e^{i\omega t}}{i\omega\sigma - \omega^2\varepsilon + \nu^2}\, d\omega = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \frac{f(\tau)\, e^{i\omega(t-\tau)}}{i\omega\sigma - \omega^2\varepsilon + \nu^2}\, d\omega\, dt, \tag{21}$$

where f_{ω} is the spectral density of the function f(t).

For the sake of simplicity, it was assumed above that the medium is described by the symmetric tensors $\hat{\varepsilon}_{ik}$, $\hat{\sigma}_{ik}$, which made it possible to simplify the writing of these tensors in the system of the principal axes. In the case of a gyrotropic medium, this would be impossible (4). However, the results preserve their form for an arbitrary tensor relation between the vectors $D_m = c^{-1}(4\pi j + \partial D/\partial t)$ and E. Assuming that $D_m = c^{-1}\hat{\varepsilon}_1' E$

or $(D_m)_{\alpha} = \frac{1}{c} \sum_{k} \hat{\varepsilon}_{1\alpha k} E_k$, where

$$\hat{\varepsilon}_{1\alpha k} = \frac{\partial}{\partial t}\hat{\varepsilon}_{\alpha k} + 4\pi\hat{\sigma}_{\alpha k},$$

we get the following equation for $q_{\lambda i}$:

$$\sum_{\alpha k} (a_{\lambda i}^{*})_{\alpha} (a_{\lambda i})_{k} \hat{\varepsilon}_{1\alpha k} \dot{q}_{\lambda i} + \nu_{\lambda i}^{2} q_{\lambda i} = \sqrt{4\pi} \sum_{k} e_{k} (V_{k} a_{\lambda i}^{*}) e^{-ik_{\lambda} r_{k}}, \tag{22}$$

Where

$$\nu_{\lambda i}^2 = c^2 \left[k_\lambda^2 \left(a_{mn} a_{mn}^* \right) - \left(k_m a_{mn}^* \right) \left(k_m a_{mn} \right) \right].$$

3. FIELD OF AN OSCILLATOR IN AN ISOTROPIC ABSORBING MEDIUM

As an example, showing the use of Hamilton's method, we shall find the field of an electron in an isotropic, absorbing medium. Let the electron, which is located at the origin, carry out small vibrations about the z axis according to the law $z = z_0 \exp(i\nu_0 t)$. Assuming, as usual, that the factor $\exp(-ik_\lambda r_\beta)$ is approximately unity, we shall have the solution:

$$\varepsilon \ddot{q}_{\lambda i} + 4\pi\sigma \dot{q}_{\lambda i} + \frac{\nu_{\lambda i}^2}{a_{\lambda i}^2} q_{\lambda i} = i\sqrt{4\pi}\, e\, \nu_0 \frac{(z_0 a_{\lambda i})}{a_{\lambda i}^2} e^{i\nu_0 t}, \qquad (23)$$

the forced solution of which is

$$q_{\lambda i} = i \frac{\sqrt{4\pi}\, e\, \nu_0 (z_0 a_{\lambda i})}{\nu_{\lambda i}^2 - \nu_0^2 + 4\pi i \xi \nu_0} e^{i\nu_0 t}. \qquad (24)$$

Here we set

$$\xi = \sigma(\nu_0)/\varepsilon(\nu_0); \qquad \varepsilon(\nu_0)\, a_{\lambda i}^2 = 1.$$

It follows from (19) here that $\nu_{\lambda i}^2 = c^2 k_\lambda^2 \lambda_{\lambda i}^2 = c^2 k_\lambda^2 / n^2(\nu_0)$.

Substituting the solution (24) in the sum (13), we get

$$A(r,t) = 4\pi c i e^{i\nu_0 t} \sum_{\lambda i} \frac{e\,\nu_0 (z_0 a_{\lambda i})\, a_{\lambda i}}{\nu_{\lambda i}^2 - \nu_0^2 + 4\pi i \xi \nu_0} e^{ik_\lambda r}. \qquad (25)$$

We change the summation in (25) to integration in the spherical coordinates Θ, ϕ, r over the solid angle Ω, taking it into account that the number of oscillators in the frequency range $\nu_{\lambda i} - \nu_{\lambda i} + d\nu_{\lambda i}$ with normals in the solid angle $d\Omega$ is equal to

$$\frac{k_\lambda^2\, d k_\lambda}{(2\pi c)^3} = \frac{n_{\lambda i}^3(\nu_0)\, \nu_{\lambda i}^2\, d\nu_{\lambda i}}{(2\pi c)^3} d\Omega. \qquad (26)$$

In integration over the frequency, we shall not write down the symbols λi.

We compute the component A_Θ at the point whose radius vector is R_0 ($\phi = 0$, $\Theta = \beta$, $r = R_0$):

$$A_{\theta}(R_0, t) = ie^{i\nu_0 t}\frac{P_0 \nu_0 n(\nu_0)}{2\pi^2 c^2}\int_{\nu}\int_0^{\pi}\int_{\theta}^{2\pi}\frac{\sin^2\Theta(\cos\Theta\cos\varphi\cos\beta + \sin\Theta\sin\beta)}{\nu^2 - \nu_0^2 + 4\pi i\xi\nu_0}\times \tag{27}$$

$$\times e^{i\frac{n(\nu_0)}{c}R_0\nu(\sin\Theta\cos\varphi\sin\beta + \cos\Theta\cos\beta)}\, d\nu\, d\varphi\, d\Theta .$$

(P_0 = ez_0 = dipole moment).

Integrating over ϕ, we have:

$$A_{\theta}(R_0, t) = ie^{i\nu_0 t}\frac{P_0 \nu_0 n(\nu_0)}{\pi c}\int_{\nu}\frac{\nu^2 d\nu}{\nu^2 - \nu_0^2 + 4\pi i\xi\nu_0}\Big\{ i\cos\beta\int_0^{\pi}\sin^2\Theta\cos\Theta\times$$

$$\times I_1\left[\frac{n(\nu_0)}{c}R_0\nu\sin\Theta\sin\beta\right]\exp\left[in(\nu_0)c^{-1}R_0\nu\cos\Theta\cos\beta\right]d\Theta + \tag{28}$$

$$+\sin\beta\int_0^{\pi}\sin^3\Theta\, I_0\left(\frac{n}{c}R_0\nu\sin\Theta\sin\beta\right)\exp\left[i\frac{n(\nu_0)}{c}R_0\nu\cos\Theta\cos\beta\right]d\Theta\Big\} .$$

Here, I_0 and I_1 are the Bessel functions of zeroth and first order. The expression in curly brackets in (28) can be computed with the aid of the Gegenbauer integral (5):

$$\int_0^{\pi} e^{iz\cos\Theta\cos\beta}\, I_{\nu-1/2}(z\sin\Theta\sin\beta)\, C_r^{\nu}(\cos\Theta)\sin^{\nu+1/2}\Theta\, d\Theta =$$

$$= \left(\frac{2\pi}{z}\right)^{1/2} i^r \sin^{\nu-1/2}\beta\, C_r^{\nu}(\cos\beta)\, I_{\nu+r}(z) \tag{29}$$

(the meaning of the functions $C_n^{\nu}(\cos\Theta)$ is shown also in the book (5); it is equal to

$$\frac{2c\sin\beta}{nR_0\nu}\left[\left(1-\frac{c^2}{n^2R_0^2\nu^2}\right)\sin\left(\frac{nR_0\nu}{c}\right)+\frac{c}{nR_0\nu}\cos\left(\frac{nR_0\nu}{c}\right)\right].$$

Starting out with the help of the theory of ~~deductions~~ residues, we carry out the integrations over ν and obtain

$$A_\theta(R_0)=\frac{P_0\sin\beta}{R_0}e^{i(\nu_0 t-k'R_0)}\left[\frac{1}{R_0\sqrt{\varepsilon'}}+ik_0\left(1-\frac{1}{k'^2R_0^2}\right)\right]+ \\ +e^{i\nu_0 t}\frac{iP_0k_0\sin\beta}{R_0^3k'^2}, \tag{30}$$

where

$$k'=\frac{\nu_0}{c}\sqrt{\varepsilon'(\nu_0)}=k_0\sqrt{\varepsilon'(\nu_0)},\ \varepsilon'(\nu_0)=\varepsilon(\nu_0)+\frac{4\pi\sigma(\nu_0)}{i\nu_0}.$$

We find the component of the electric field

$$E_\theta(R_0)=-ik_0A_\theta(R_0)-\frac{1}{R_0}\frac{\partial\varphi}{\partial\Theta}(R_0).$$

Making use of the formula (10) for calculation of $\mathrm{grad}_\Theta\varphi$, we establish the fact that the term $(1/R_0)\,\partial\varphi/\partial\Theta$, computed approximately with account of the smallness of z_0, reduces to the term corresponding to the last component in (30), which depends only on the time. Finally, we obtain the well-known formula (see, for example, Ref. (3))

$$E_\theta(R_0)=\frac{P_0\sin\beta}{\varepsilon'R_0}\left[k'^2-\frac{1}{R_0}\left(ik'-\frac{1}{R_0}\right)\right].$$

In conclusion, I express my gratitude to V. V. Zheleznyakov, whose advice I employed in the writing up of the work.

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RADIATION OF FERROMAGNETIC SYSTEMS

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(abstract) The problem is considered of the radiation of ferromagnetic systems close to the compensation point.

It was shown in Refs. (1-3) that in adiabatic magnetic polarity reversal of a ferromagnetic or paramagnetic specimen, a coherent spontaneous radiation is produced with a frequency corresponding to ferromagnetic (paramagnetic) resonance.

It is of interest to consider the problem of the behavior of ferromagnetic systems in the presence of an external variable magnetic field. As is seen from what follows, it is possible, using such systems, to obtain radiation with a frequency larger than that which is obtained in magnetic reversal of an ordinary ferromagnet. Thus, ferromagnetic systems can be of interest from the viewpoint of the utilization of much higher frequencies.

1. INITIAL EQUATIONS

If the ferromagnet consists of two sublattices with magnetization vectors M_1 and M_2, then the equations of motion will have the form:

$$\dot{\boldsymbol{M}}_1 = \gamma_1 [\boldsymbol{M}_1, \boldsymbol{H}_1]; \quad \dot{\boldsymbol{M}}_2 = \gamma_2 [\boldsymbol{M}_2, \boldsymbol{H}_2], \tag{1}$$

where $\boldsymbol{H}_1 = \boldsymbol{H}_0 + \boldsymbol{H}_{1A} + \boldsymbol{H}_{1E}, \quad \boldsymbol{H}_2 = \boldsymbol{H}_0 + \boldsymbol{H}_{2A} + \boldsymbol{H}_{2E}$

(H_0 is the external magnetic field, H_{1A}, H_{2A} are the anisotropic fields acting in the first and second sublattices, respectively, and H_{1E} and H_{2E} are the exchange force fields, γ_1, γ_2 are the gyro-

magnetic ratios).

Damping is not taken into account in equations (1). Account of radiation damping is made in the same fashion as was down in the work of Ginzburg (4) (see also (5)). As a result, it is shown that the external magnetic field H_0 must be replaced by

$$H = H_0 - \frac{4\omega_m}{3\pi c^3}\ddot{\mu} + \frac{2}{3c^3}\dddot{\mu} = H_0 - \frac{4\omega_m V}{3\pi c^3}\ddot{M} + \frac{2V}{3c^3}\dddot{M}, \quad (2)$$

where $\mu = \int_V (M_1 + M_2)dV = MV$ is the total magnetic moment of the specimen, $\omega_m \simeq c/R$, where R is the radius of the specimen, or a quantity of the order of the linear dimensions of the specimen. The third term in the expression (2) gives the dissipation. The second term gives rise to a certain frequency shift and is conservative (5). In the present work, we shall not take this term into account.

We first consider in more detail the case without account of the anisotropic field.

Making use of the fact that the fields of exchange forces have the form

$$H_{1E} = \lambda M_2; \quad H_{2E} = \lambda M_1$$

(where λ is a constant of the molecular field), equation (1) can (in the case of $H_{1A} = H_{2A} = 0$) be written in the form:

$$\dot{M}_1 = \gamma_1 [M_1, H + \lambda M_2]; \quad \dot{M}_2 = \gamma_2 [M_2, H + \lambda M_1]. \quad (3)$$

If we transform to new variables $M = M_1 + M_2$ and $S = M_1/\gamma_1 + M_2/\gamma_2 \equiv S_1 + S_2$, which have the meaning of the total magnetization and the density of the total spin of the system, then equation (3) can be written in the form:

$$\dot{M} = -\gamma_1\gamma_2 \left[S, H_0 + \lambda M + \frac{2V}{3c^3}\ddot{M} \right] + (\gamma_1 + \gamma_2)\dot{S}; \quad (4)$$

$$\dot{S} = \left[M, H_0 + \frac{2V}{3c^3}\ddot{M} \right]. \quad (5)$$

It is not difficult to show that equations (4)-(5) have the following integrals of the motion:

$$MS - \gamma_0 S^2 = C_1; \quad M^2 - \gamma_1\gamma_2 S^2 = C_2 \quad (6)$$

or

$$M_1 = \text{const}; \quad M_2 = \text{const}; \quad (6a)$$

$$S_1 = \text{const}; \quad S_2 = \text{const},$$

where $\gamma_0 = (\gamma_1 + \gamma_2)/2$.

Without consideration of dissipation, the energy integral is

$$U = -(H_0 M) - \frac{\gamma_1 \gamma_2 \lambda}{2} S^2. \qquad (7)$$

In what follows, we shall need equations (4)-(5) in dimensionless form. We make the following change of variables:

$$\tau = \omega_0 t; \quad \omega_0 = |\gamma H_0|; \quad H_0 = H_0 h; \quad M = M_0 m;$$

$$S = \tilde{S}_0 s; \quad \tilde{M}_0 = |M_1| + |M_2|; \quad \tilde{S}_0 = -(|S_1| + |S_2|);$$

$$|\gamma| = \frac{|M_0|}{|\tilde{S}_0|}; \quad \frac{d}{dt} = \omega_0 \frac{d}{d\tau}; \quad \frac{d^3}{dt^3} = \omega_0^3 \frac{d^3}{d\tau^3}.$$

In the new dimensionless variables, equations (4)-(5) take the form:

$$\dot{m} = -[s, \ \alpha_1 m + \alpha_2 h + \alpha_3 \dddot{m}] + \alpha_4 \dot{s}; \qquad (8)$$

$$\dot{s} = -[m, \ h + (\alpha_3/\alpha_2) \dddot{m}], \qquad (9)$$

where $\alpha_1 = \gamma_1 \gamma_2 \lambda S_0/\omega_0$; $\alpha_2 = \gamma_1 \gamma_2 \tilde{S}_0/|\gamma| \tilde{M}_0 = -\gamma_1 \gamma_2/|\gamma|^2$;

$$\alpha_3 = 2\gamma_1 \gamma_2 \tilde{S}_0 \omega_0^2 V/3c^3; \quad \alpha_4 = -(\gamma_1 + \gamma_2)/|\gamma|.$$

2. THE RADIATION PROCESS IN THE ABSENCE OF A MAGNETIC FIELD

We now consider the case which has, perhaps, only methodological interest. This is the case in which there is no external magnetic field, but the system is excited, so that it is capable of radiation.

Without dissipation, and in the absence of a magnetic field, equations (4)-(5) take the form:

$$\dot{M} - \gamma_1 \gamma_2 \lambda [S, M]; \quad \dot{S} = 0. \qquad (10)$$

These equations describe the precession of M relative to the constant spin vector, with frequency

$$\omega_e = \gamma_1 \gamma_2 \lambda S. \qquad (11)$$

It is easy to obtain the same frequency from quantum considerations. In the absence of a magnetic field, the Hamiltonian of the system has the form:

$$H = -\lambda M_1 M_2 = \lambda\gamma_1\gamma_2 S_1 S_2 = \frac{\lambda\gamma_1\gamma_2}{2}\{S(S+\hbar) - S_1(S_1+\hbar) - S_2(S_2+\hbar)\}. \quad (12)$$

The frequency of the radiation, corresponding to the transitions $\Delta S = \hbar$ ($\Delta S_1 = \Delta S_2 = 0$), is equal to

$$\omega = \frac{H_{S-\hbar} - H_S}{\hbar} = -\frac{\gamma_1\gamma_2\lambda}{2\hbar}\{(S-\hbar)S - S(S+\hbar)\} = \gamma_1\gamma_2\lambda S.$$

Radiation retardation can be obtained from energy considerations

$$-\frac{dH}{dt} = \frac{2}{3c^3}(\ddot{M})^2. \quad (13)$$

Thissame equation can be obtained from (4), (5) (for $H_o = 0$, by assuming the damping to be small. On the right-hand side of (13), we substitute $\ddot{M}$ from equation (10) (here we neglect terms of order $1/c^3$, which is equivalent to the neglect of terms of the order $(1/c^3)^2$ in equation (13). Then

$$\ddot{M} = \gamma_1\gamma_2\lambda[\dot{M}, S] = (\gamma_1\gamma_2\lambda)^2[[M, S]S] = -(\gamma_1\gamma_2\lambda)^2\{MS^2 - S(MS)\};$$

$$(\ddot{M})^2 = (\gamma_1\gamma_2\lambda)^4\{M^2S^4 - S^2(MS)^2\}.$$

Further, we make use of the integrals (6):

$$M^2 = C_2 + \gamma_1\gamma_2 S^2; \quad (MS) = C_1 + \gamma_0 S^2.$$

After substitution, we obtain:

$$(\ddot{M})^2 = (\gamma_1\gamma_2\lambda)^4\{-C_1 S^2 + (C_2 - 2C_1\gamma_0)S^4 + (\gamma_1\gamma_2 - \gamma_0^2)S^6\}.$$

After setting $S^2 = u$, equation (13) takes the form:

$$\frac{du}{dt} = \frac{4(\gamma_1\gamma_2\lambda)^3}{3c^3}\{-C_1^2 u + (C_2 - 2C_1\gamma_0)u^2 + (\gamma_1\gamma_2 - \gamma_0^2)u^3\}. \quad (14)$$

This equation describes the process of radiation in the excitation of the ferromagnet by the exchange frequency. As is seen from (6), (11) and (14), in addition to the change of angle between M and S, a change in frequency takes place which is proportional to $\sqrt{u}$.

We shall not consider here any further analysis of equation (14)(which is easily solved in terms of elementary functions), since

the capability of excitation of radiation at the frequency ω_e is not known.*

3. MAGNETIC POLARITY REVERSAL NEAR THE COMPENSATION POINT

If use is made of equations (1), then in linear approximation, when the magnetic moment is almost parallel (antiparallel) to the magnetic field H, one can obtain the following characteristic frequencies of the system (6):

$$\omega = \gamma_0 H_0 + \delta H_A + \frac{1}{2}\lambda\gamma_1\gamma_2 S \pm \Big\{(\gamma_0 H_A + \delta H_0) \mid \gamma_0 H_A + \delta H_0 - \lambda\gamma_1\gamma_2 (S_1 - S_2) \mid + \frac{1}{4}\lambda^2\gamma_1^2\gamma_2^2 S^2\Big\}^{1/2}, \quad (15)$$

where $\delta = (\gamma_1 - \gamma_2)/2$, $H_A = |H_{1A}| = -|H_{2A}|$.

We now consider this expression close to the point of compensation $S = S_1 + S_2 = 0$. In this case, expression (15) takes the form:

$$\omega = \gamma_0 H_0 + \delta H_A \pm \sqrt{(\gamma H_A + \delta H_0) \mid \gamma_0 H_A + \delta H_0 + 2\lambda\gamma_1\gamma_2 S_2 \mid} \simeq \gamma_0 H_0 + \delta H_A \pm \sqrt{(\gamma_0 H_A + \delta H_0) H_E}, \quad (16)$$

where $H_E = 2\lambda\gamma_1\gamma_2 S_2/\gamma_0$ is the intensity of the molecular field (the field of the exchange forces) and $|H_E| \gg |\gamma_0 H_A + \delta H_0|$, which is the usual case.

It is evident from equation (16) that upon inversion of the magnetic field, the radicand becomes negative if the condition

$$|\gamma_0 H_A| < |\delta H_0|. \quad (17)$$

is fulfilled. Then the system becomes unstable, and the components of the magnetic moment begin to increase exponentially.

We note that in the inversion of the field in a c (1-3), the components of the magnetic moment also increase. However, there this is connected with the dissipative mechanism; here we have not yet considered dissipation. Such an increase is brought about only by the conservative terms, which are of a much larger order of magnitude than the dissipative; therefore, the increase takes place much more rapidly. In fact, the exponent is of the order of

* If, of course, one does not consider the trivial capability of radiation with the same frequency ω_e.

$$\sqrt{(\gamma_0 H_A + \delta H_0)\gamma_0 H_E} \equiv \gamma H_{\text{экв}}.$$

For $H_A + (\delta/\gamma_0) \simeq 10^3$ oersteds and $H_E \simeq 10^7$ oersteds, the equivalent magnetic field is $H_{eq} \simeq 10^5$ oersteds.

For such a rapid growth, the applicability of the linear approximation is rapidly destroyed. Therefore, it is necessary to find the solution of the corresponding nonlinear equations. Numerical solution of equations (8)-(9) (without the anisotropic field) has been carried out. Since $H_A = 0$ in these equations, then the increase in the magnetic moment is obtained immediately after the change in sign of H_0. The results of the numerical calculation are given in Figs. 1-7.

In the solution of equations (8)-(9), the following values were used for the parameters and the initial conditions (Figs. 1,2): $\alpha_1 = H_E/H_0 = 1280$; $\alpha_2 = -1.5$; $\alpha_3 = 0$; $\alpha_y = 2.5$; $h = (0, \sin\beta t, 0)$; $\beta = 1$; $m_3 = s_3 = 2^{-7}$ (it is necessary to furnish some sort of sufficiently small arbitrary perturbation);

$$m_2 = -|\gamma_1 - \gamma_2|/2|\gamma| = -\sqrt{\alpha_1^2 + 4\alpha_2}/2 = -0{,}25;\ s_2 = 0;\ m_1 = s_1 = 0.$$

Dissipation is not considered in this solution.

Numerical solutions with account of dissipation were carried out for the same initial conditions and values of the parameters (except α_3, which determines the dissipation, and which is equal to -2^{-21}, -2^{-17}, -2^{-15}, -2^{-14}, respectively)*. Graphs with $\alpha_3 = -2^{-14}$ are shown in Figs. 3 and 5. We note that the systematic increase in the dissipation scarcely shows in the front of the pulse m_1, m_2, m_3, which arises in the change of sign of the magnetic field. Account of dissipation leads to an increased damping of the pulse after it appears.

Thus, as a result of the action on the ferromagnet near the point of compensation of the variable field

$$H_y = H_0 h_2 \sin(\beta t), \quad H_x = H_z = 0$$

a variable magnetic moment arises, with a frequency determined by the ratio $\alpha_1 = H_E/H_0$. This frequency, as is seen from the graphs, is at least larger in order of magnitude than the frequency $\omega_0 = |\gamma H_0|$. The amplitude of the magnetic moment is of the order $M_0 m_2 = M_0 |\gamma_1 - \gamma_2|/2|\gamma|$. Dissipation leads to damping of the pulse within the time of magnetic reversal. The frequency is almost independent of β (see Fig. 5 for $\beta = \frac{1}{2}$) and increases for increase in H_0 (see Fig. 6). The relative frequency, measured in units of $\omega_0 = |\gamma H_0|$ does decrease; however, as is not difficult to see, the frequency unit increases more rapidly than the relative frequency decreases.

*The latter value of α_3 corresponds to real radiation damping of the sample in the volume $\sim(\lambda/4)^3$, where $\lambda \sim$ wavelength radiated (from graphs).

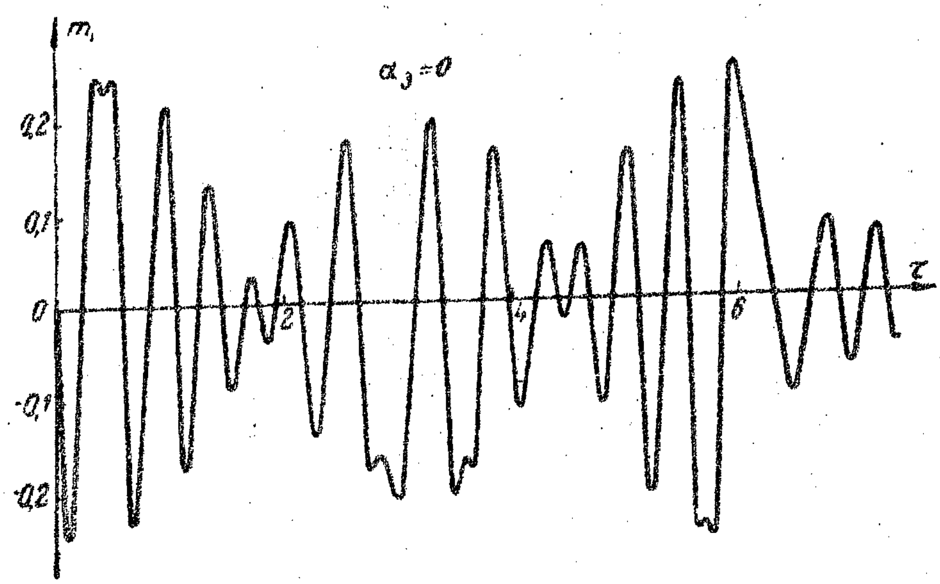


Fig. 1

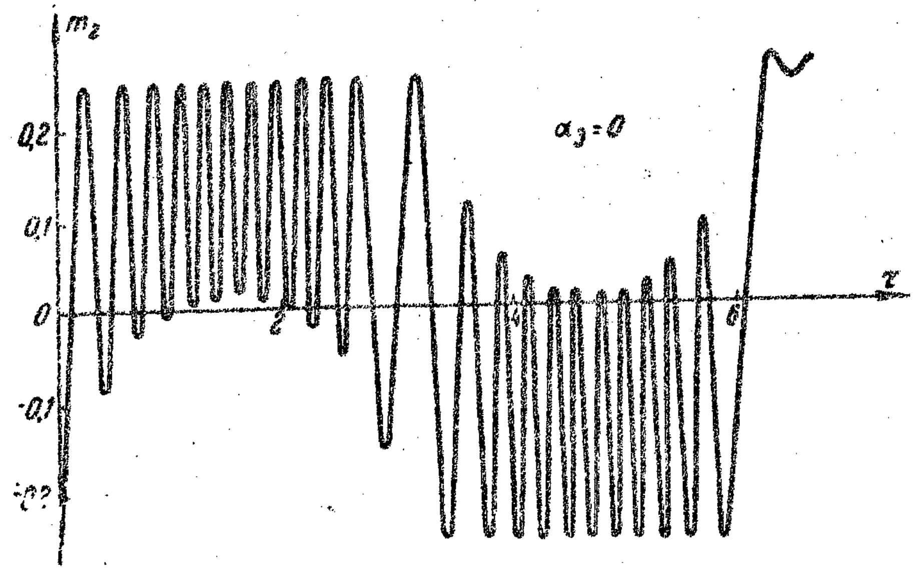
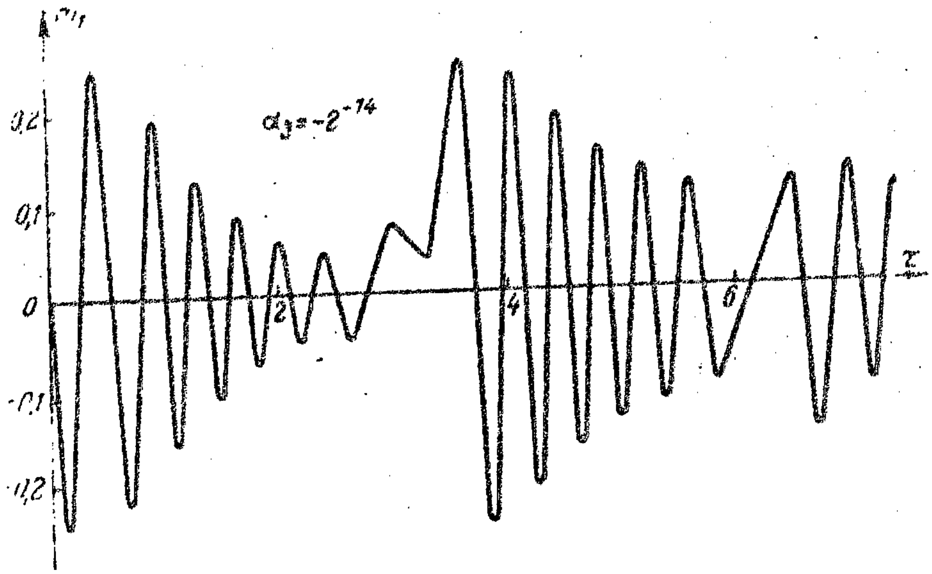


Fig. 2

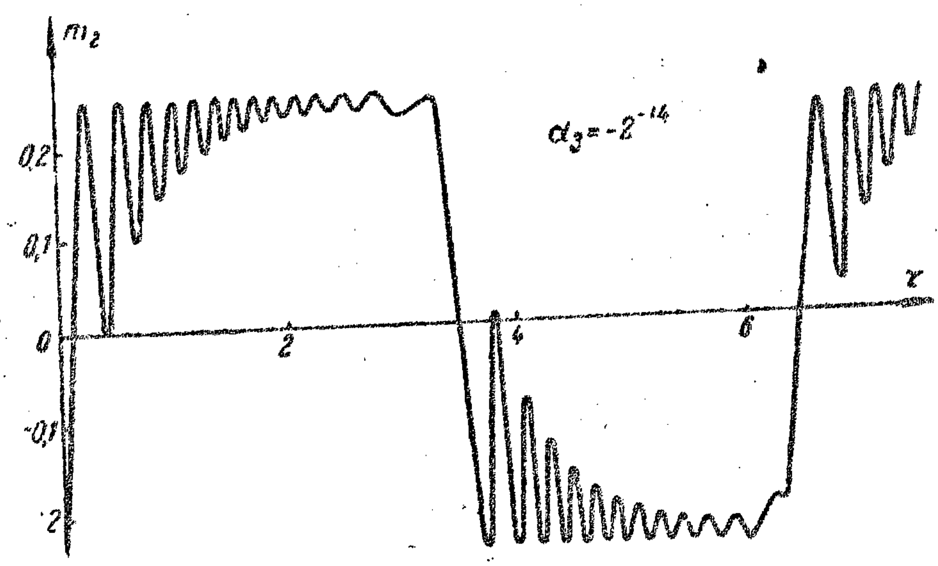
Here not very realistic values of $|\gamma_1 - \gamma_2|/2|\gamma|$ or α_1 and α_4 are used. However, this was done in order to demonstrate the effect more clearly.

A curve with a dissipation term of the form α_3 m (non-radiative) for $\alpha_3 = 1/10$ is shown in Fig. 7. In order of magnitude, this damping is much greater than that of the previous graphs.



m_1
$\alpha_3 = -2^{-14}$
0,2
0,1
0
-0,1
-0,2
2
4
6
τ

Fig. 3.



m_2
$\alpha_3 = -2^{-14}$
0,2
0,1
0
-0,1
-2
2
4
6
τ

Fig. 4.

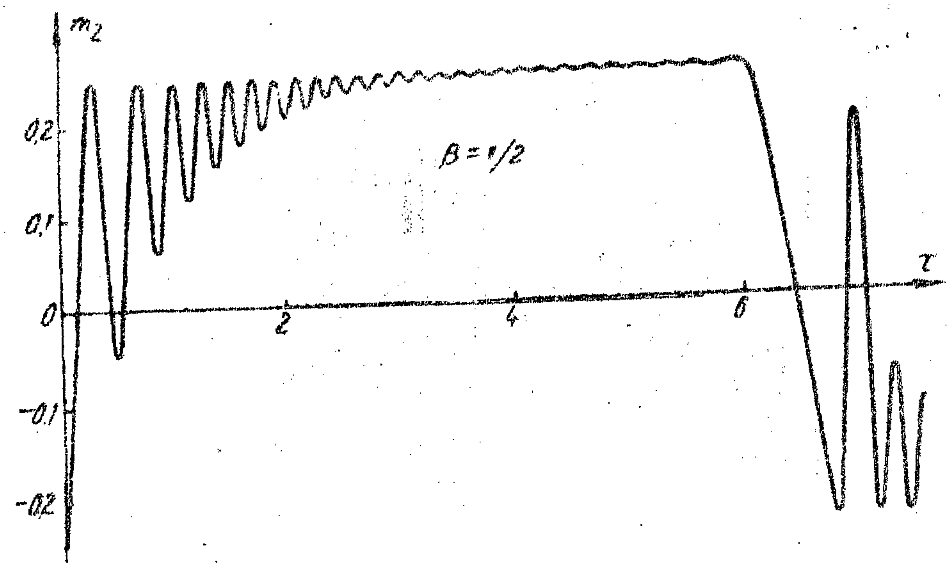


Fig. 5.

Use of ferromagnets with points of compensation can lead to a number of new and interesting effects. One of these, considered in the present work, consists in the creation of a pulse with a very steep front, full of high frequency. In fact, the large amplitude of all

three components arises immediately and for any damping. Since the radiation in the dipole approximation (when the dimensions of the system are much smaller than the radiated wavelengths) is determined by the magnetic moment of the system, then, along with the pulse of the magnetic moment, there also arises a pulse of the radiation field of the same form. It is of interest to note that here (in contrast to the ordinary cohetron) the component of the magnetic moment also oscillates with high frequency (the component parallel to the magnetic field*). Thus the effect considered here can serve to obtain pulses of the electromagnetic field with a very steep front and very high frequency.

We note that one can obtain the radiation not only in the form

* As is evident from the graphs, with approximately doubled frequency.

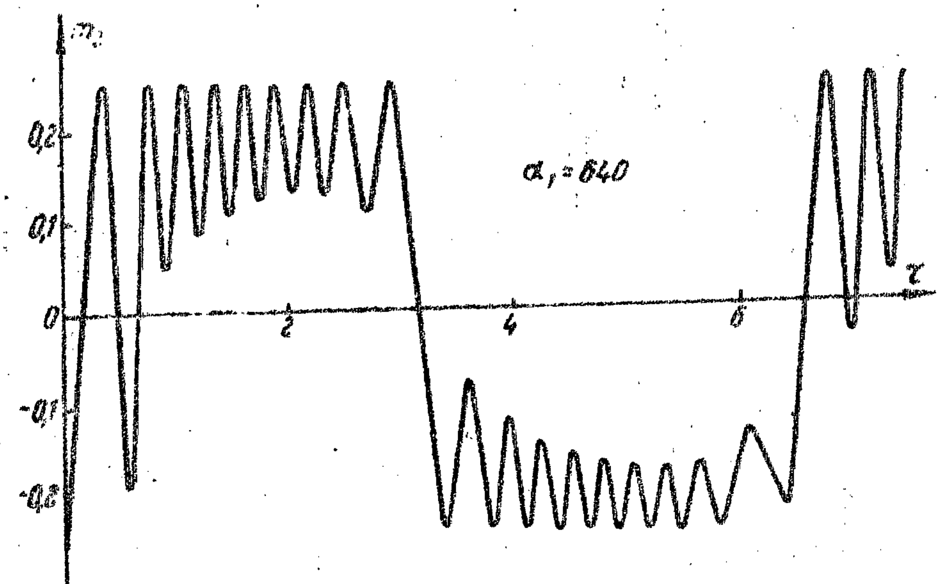


Fig. 6.

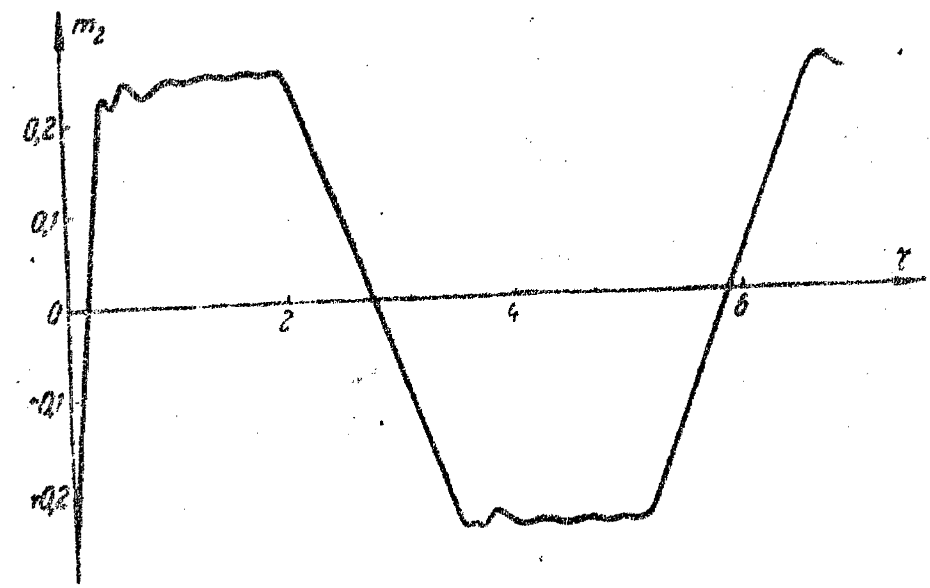


Fig. 7.

of pulses, but also continuously, with the amplitude modulated with the frequency of the given pulse β . This is also evident from the graphs.

Shock electromagnetic waves can serve as another possible application of ferrites with a compensation point.* Here, it is natural to expect the appearance of the same high frequency oscillations on the front of the shokk wave as we have just been discussing.

In conclusion, I want to express my deep gratitude to S. I. Al'ber and T. N. Pigolkin, who carried out the numerical solution of equations (8), (9) on the computing machine M-2 of the Institute of Electronic Computing Machines, Academy of Sciences, USSR.

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* On shock electromagnetic waves in ordinary ferrites, see Ref. 7

THE PHENOMENOLOGICAL THEORY OF FREE PRECESSION OF MAGNETIC MOMENTS OF ATOMIC NUCLEI

[This is a translation of an article written by V. M. Ryzhkov, G. V. Skrotskii and Yu. I. Alimov in Radiofizika (Radiophysics) Vol. II, No. 6, 1959, pages 884-891.]

(Abstract) The conditions for obtaining the free precession of the vector of the nuclear magnetization of a specimen by the method of Packard and Varian are discussed. The effect of the inhomogeneity of the measured field on the damping of the signal of free precession is considered.

INTRODUCTION

The phenomenon of free precession of the vector of the nuclear magnetization, predicted by Bloch [1], was discovered experimentally in the earth's magnetic field by Packard and Varian [2]. Recently it has found application in the technology of geomagnetic measurement [3-8] and it is beginning to be used for the study of the physical properties and structure of matter [9-10]. The theory of this phenomenon is acquiring practical importance at the present time.

The essence of the method of Packard-Varian consists in the following. A specimen of the material with a long relaxation time (of the order of a second) is magnetized by a sufficiently auxiliary field H_0 in a direction perpendicular to the weak measured field h_0. In a rapid switching off of the auxiliary field, the magnetization vector $\underset{\sim}{M}$ fails to change its value and its orientation relative to the field h_0, and begins to precess around it with frequency $\omega_0 = \gamma h_0$. The coefficient γ is practically equal to the gyromagnetic ratio of the nuclei under investigation [11].

For the practically important case in which transient processes connected with the switching off of the auxiliary field decay in time that is significantly smaller than the relaxation time of a system of nuclear spins, the motion of the magnetization vector in the period of duration of these transient processes must be described by the equation:

$$\dot{M} = \gamma\,|MH|. \qquad (1)$$

In this case the damping brought about by relaxation processes is neglected as unimportant in the course of this interval of time. After completion of the transient processes, when only the weak field h_0 remains, i.e., free precession of the magnetization vector takes place, we can consider the following equation to be valid [12]:

$$\dot{M} = \gamma\,|MH| + i\,\frac{\chi_0 H_x - M_x}{T_\perp} + j\,\frac{\chi_0 H_y - M_y}{T} + k\,\frac{\chi_0 H_z - M_z}{T_\perp}, \qquad (2)$$

which takes into account the relaxation processes in the system of nuclear spins. Here χ_0 is the static nuclear magnetic susceptibility of the specimen, and $T_{\parallel}$ and $T_{\perp}$ are the longitudinal and transverse relaxation times, characterizing the rate of establishment of equilibrium of the state of magnetization $\mathbf{M}$. For liquids usually used in experiments on free precession in weak magnetic fields, one can assume $T_{\parallel} = T_{\perp} = \tau$ [11]. In this case, the solution of Eq. (2) for a field $\mathbf{H}$ which is constant in magnitude and direction:

$$\boldsymbol{M}(t) = \chi_0 \boldsymbol{H} + |\boldsymbol{H}(\boldsymbol{H}\boldsymbol{M}(0))H^{-2} - \chi_0 \boldsymbol{H} + |\boldsymbol{M}(0)\boldsymbol{H}|H^{-1}\sin(\overline{\omega}t) + \\ + [\boldsymbol{H}[\boldsymbol{M}(0)\boldsymbol{H}]]H^{-2}\cos(\overline{\omega}t)|\, e^{-t/\tau}, \qquad (3)$$

where $\overline{\omega} = \gamma H$.

Such an approach to a solution of the problem set forth makes it possible to consider independently the effect of process of switching off the auxiliary field H_0 on the initial amplitude of precession and on the characteristics of the damping of the signal of free precession.

1. EFFECT OF THE PROCESS OF SWITCHING OFF THE AUXILIARY FIELD ON THE AMPLITUDE OF THE FREE PRECESSION SIGNAL

The initial amplitude of precession in the Packard-Varian method reaches a maximum for instantaneous switching off the auxiliary magnetizing field: $\mathbf{H}\ \{H_0, 0, h_0\}$ for $t < 0$, $\mathbf{H}\ \{0, 0, h_0\}$ for $t > 0$. In this case, we get from (3) for the initial

condition $\underset{\sim}{M}(0) = \chi_o \underset{\sim}{H}(0)$: $M_x = \chi_0 H_0 \exp(-t/\tau)\cos(\omega_0 t);$

$$M_y = -\chi_0 H_0 \exp(-t/\tau)\sin(\omega_0 t);$$
$$M_z = \chi_0 h_0. \tag{4}$$

The equations obtained describe the free precession of the magnetization vector $\underset{\sim}{M}$ around the field $\underset{\sim}{h}_0$, being damped in time. The initial amplitude of the emf found in a coil surrounding the specimen, for an ideally homogeneous field h_0, will in this case be equal to

$$\mathcal{E} = k\chi_0 H_0 \omega_0, \tag{5}$$

where k is a coefficient depending on the structure of the coil; the axis of the coil is directed along X.

In practice the auxiliary field H_0 is produced by passage of a constant current through the inductance coil surrounding the specimen under study. Upon breaking the magnetizing circuits (after the disappearance of the electric arc on the contacts of the key) a transient process takes place in the coil, and the change in the auxiliary field with time can be represented in the form:

$$H = H_0 e^{-\alpha t}\cos(\omega t),$$

where α is the damping constant, and ω is the frequency of the transient process. Thus the problem reduces to a solution of Eq. (1) for a magnetic field of the form:

$$H\{H_0 e^{-\alpha t}\cos(\omega t), 0, h_0\} \tag{6}$$

for $H_0 \gg h_0$.

The system of equations (1) can either be put in the form of

a Riccati equation by means of the substitution of Darboux [13], or can be transformed to an integral equation of the Volterra type [14]. Unfortunately, in the problem under consideration, it is not possible to isolate a small parameter, in connection with which the solution of the equation over the whole range of values $t \geq 0$ can be obtained in the form of a slowly converging series, which is ill-suited for application. Therefore, in what follows we limit ourselves only to the qualitative estimate of the effect of the switching-off process which can be done, and do not seek the solution of Eqs. (1) for the entire region $t \geq 0$.

For explanation of the characteristic features of the problem under consideration, we shall choose the frequent case in which the magnetic field intensity vector $\underset{\sim}{H}$ rotates with constant angular velocity Ω to an angle of 90°:

$$H\{H_0 \cos(\Omega t),\ 0,\ H_0 \sin(\Omega t)\} \qquad \text{for} \quad 0 \quad t \quad \pi/2\Omega;$$

$$H\{0,\ 0,\ H_0\} \qquad \text{for} \quad t \quad \pi\ 2\Omega. \qquad (7)$$

In the system of coordinates X'YZ' (Fig. 1), which rotates with angular velocity Ω, the magnetic field (7) is transformed to the form: $\underset{\sim}{H}\ \{H_0,\ -\Omega/\gamma,\ 0\}$. Solution of Eq. (1) in this system of coordinates, under the condition that at the time $t = 0$ the vector $\underset{\sim}{M}(0)$ has components $\chi_0 H_0, 0, 0$, has the form:

$$M'_x = \chi_0 H_0 \{H_0^2 H^2 + (\Omega^2/\gamma^2 H^2)\cos(\bar{\omega} t)\};$$

$$M'_y = \chi_0 H_0 \{-(\Omega H_0/\gamma H^2)\,[1 - \cos(\bar{\omega} t)]\};$$

$$M'_z = -\chi_0 H_0 (\Omega/\gamma H)\sin(\bar{\omega} t),$$

where $\bar{\omega} = \gamma H$; $H = \left[H_0^2 + (\Omega/\gamma)^2\right]^{1/2}$.

Transforming to the original system of coordinates we find for $t_0 = \pi/2\Omega$:

$$M_x = (\chi_0 H_0 \Omega/\gamma H) \sin(\pi\bar{\omega}/2\Omega);$$

$$M_y = -(\chi_0 H_0^2 \Omega/\gamma H^2)\,[1 - \cos(\pi\omega/2\Omega)];$$

$$M_z = \chi_0 H_0\,[H_0^2 H^2 + (\Omega^2/\gamma^2 H^2)\cos(\pi\omega\, 2\Omega)]. \quad (8)$$

Now, using Eqs. (8) as initial conditions, we have for $t > t_0$:

$$M_x = A \sin(\Omega_0 t + \varphi);$$

$$M_y = A \cos(\Omega_0 t + \varphi);$$

$$M_z = M_z. \quad (9)$$

where $\Omega_0 = \gamma H_0$, $A = [(M_x)^2 + (M_y)^2]^{1/2}$ and $\phi = \arctan(M_x/M_y)$. The equations (9) describe the free precession of the magnetization vector around the field H_0 parallel to the Z axis. Using (8) and (9) we obtain the following expression for the amplitude of the emf generated in the coil surrounding the specimen

$$\mathcal{E} = k\chi_0 H_0 \Omega_0 \left[1 - \frac{1 + x^2\cos(\pi\sqrt{1 + x^2}/2x)}{1 + x^2}\right]^{1/2},$$

where $x = \Omega/\Omega_0$.

The dependence of the amplitude of the precession signal $\mathcal{E}/k\chi_0 H_0\Omega_0$ on the angular velocity of rotation of the magnetic field vector, Ω/Ω_0, is shown graphically in Fig. 2. As is seen from the graph, for $\Omega < 0.5\Omega_0$, the magnetization vector virtually follows the direction of the field in its rotation (adiabatic case). For $\Omega > \Omega_0$, the magnetization vector is virtually unable to change its orientation (non-adiabatic case). Thus, to obtain a large initial amplitude of the signal of the free precession, there must be a turning of the vector of the magnetic field intensity H in a time less

than the period of Larmor precession in this field.

To explain the effects of transient vibrations of the switching field, we consider another special case, in which there is only one oscillating field $\underset{\sim}{H}$ $\{H_0 \cos(\omega t), 0, 0\}$. The solution of the set of equations (1) for the initial conditions $\underset{\sim}{M}(0)$ $\{M_x(0), 0, M_z(0)\}$ in this case has the form:
$$M_x = M_x(0);$$

$$M_y = M_y(0) \sin |(\Omega_0/\omega) \sin(\omega t)|;$$
$$M_z = M_z(0) \cos |(\Omega_0/\omega) \sin(\omega t)|.$$

The resultant equations describe the precession of the magnetization vector around the direction of the oscillating field with variable frequency. The component of the vector $\underset{\sim}{M}$ in the direction of the field in this case does not change. For $\Omega_0 << \omega$, the equations will describe small oscillations of the magnetization vector about the initial position. Thus, with accuracy up to these small oscillations, the oscillating field has no effect on the motion of the magnetization vector if its angular velocity is much greater than the velocity of precession Ω_0.

For the case in which the external field has the form (6), we shall assume that the effect of the transient process reduces to a small perturbation of the solution of Eq. (1) obtained for instantaneous switching. Solution of Eq. (1) for the components M_x and M_y in this case can be represented in complex form:

$$M_{xy} = M_{xy}(0)\, e^{i\omega_0 t} + i\,\Omega_0\, e^{i\omega_0 t} \int_0^t M_z(t)\, e^{(i\omega_0 - \alpha)t} \cos(\omega t)\, dt, \tag{10}$$

where $M_{xy} = M_x + iM_y$. The first term on the right hand side of this expression represents the solution of Eq. (1) for the case of instantaneous switching of the field H_0. The second term determines the perturbation brought about by the transient process upon switching off of the field H_0. If we set $M_z(t) \simeq M_z(0)$ in first approximation, then by computing the integral in (10) and neglecting damping terms, we get, for $t \gg 1/\alpha$:

$$M_{xy} = M_{xy}(0)\, e^{-i\omega_0 t} + i\,\Omega_0 M_z(0)\,(\xi + i\eta)\, e^{i\omega_0 t}, \tag{11}$$

where

$$\xi = \frac{\alpha\,(\alpha^2 + \omega_0^2 + \omega^2)}{[\alpha^2 + (\omega_0 + \omega)^2]\,[\alpha^2 + (\omega_0 - \omega)^2]};$$

$$\eta = \frac{\omega_0\,(\alpha^2 + \omega_0^2 - \omega^2)}{[\alpha^2 + (\omega_0 + \omega)^2]\,[\alpha^2 + (\omega_0 - \omega)^2]}. \tag{12}$$

for the initial conditions $\underset{\sim}{M}(0)\ \{\chi_0 H_0,\ 0,\ \chi_0 h_0\}$ we get from (11):

$$M_x = B \sin(\omega_0 t + \psi); \quad M_y = B \cos(\omega_0 t + \psi),$$

where

$$B = \chi_0 H_0\, |(\omega_0\xi)^2 + (1 - \omega_0\eta)^2|^{1/2};$$

$$\psi \quad \operatorname{arctg} \frac{1 - \omega_0\eta}{\omega_0\xi}. \tag{13}$$

Assuming the quantities $\omega_0\xi$ and $\omega_0\eta$ to be small in comparison with unity, and neglecting their squares in (13), we get for the amplitude of the emf generated in the coil whose axis coincides with the X axis,

$$\mathcal{E} = k\chi_0 H_0 \omega_0 (1 - \omega_0 \eta).$$

This expression differs from (5) by the factor $(1 - \omega_0 \eta)$, which is close to unity for $\omega_0 \eta \ll 0.5$. For this case, by using (12), we find:

$$\alpha^2 \gg -\omega^2 \pm \omega_0^2.$$

Taking it into account that $\alpha \geq 0$, we obtain the conditions which α and ω must satisfy in order that the perturbation be small (i.e., that the amplitude of the emf of the precession for a decrease of the magnetizing field according to the law $H_0 \exp(-\alpha p) \cos(\omega t)$ differs slightly from the amplitude of the emf for instantaneous switching of the field H_0):

$$\begin{aligned} \alpha &\geqslant 0 \quad \text{if} \quad \omega > \omega_0; \\ \alpha &\gg \omega_0 \quad \text{if} \quad \omega < \omega_0. \end{aligned} \tag{14}$$

Thus, for a frequency of the transient process ω, significantly larger than the frequency of precession ω_0 in the measured field h_0, the transient process has practically no effect on the initial amplitude of the signal of the precession, even in the absence of damping ($\alpha = 0$). This conclusion is identical with what was obtained earlier for the special case in which there was only one oscillating field. For a frequency of the transient process ω less than the frequency of precession ω_0, the transient process will have no effect on the initial amplitude of the signal of the precession only for the case of its rapid attenuation. In particular, for a periodic decrease of the auxiliary field ($\omega = 0$), the damping constant α must be much

larger than the frequency of precession ω_0.

2. EFFECTS OF INHOMOGENEITY OF THE MEASURED FIELD ON THE DAMPING OF THE SIGNAL OF THE FREE PRECESSION

If we neglect the effect of the circuit of the detecting coil on the damping of the free precession [15], then, for an ideally homogeneous field h_0, the electromotive force induced in the coil will, in accord with (4), decay exponentially with time constant τ. The effect of inhomogeneity of the measured field from the macroscopic point of view is that the frequencies of precession of the magnetic moments of the different elements of volume of the specimen will be different, and they will diverge in phase as time passes, which leads to an additional damping of the precession. This effect was investigated experimentally by Waters [16]; in his opinion, it could be reduced to a decrease in the time constant of the attenuation, which in this case (by analogy with the effect in nuclear magnetic resonance) is designated in [16] by T_2. However, it was shown earlier [17] that the form of the nuclear resonance signal for "rapid passage" depends materially on the form of the distribution function of the external field in the specimen, and in the case of a constant gradient within the limits of the sample "beats" can be observed [18]. In the case of "rapid passage" of narrow resonance lines, conditions are obtained which are close to the conditions of free precession; therefore, one can expect in the case of a constant gradient of a magnetic field within the limits of the specimen that

there will be "beats" of the signal of free precession.

Let us consider the effect of the inhomogeneity of the measured field on the damping of the precession signal for the case of a cylindrical specimen, which is of practical importance (Fig. 3). We shall assume that the magnetic field within the limits of the specimen changes linearly: $H \{ Gx/2, Gy/2, h_0 - Gz \}$ and $2GR_0/h_0 \ll 1$. Assuming that the specimen was homogeneously magnetized at the initial instant of time: $\underset{\sim}{M}(0) \{ \chi_0 H_0, 0, \chi_0 h_0 \}$, and that the magnetizing field H_0 was instantaneously shut off at $t = 0$, and further taking it into account that $H_0 \gg h_0$; $\bar{\omega} \gg 1/\tau$, we get from (3):

$$\dot{M}_x = -\chi_0 H_0 \exp(-t/\tau)\, \bar{\omega} \sin(\bar{\omega} t), \tag{15}$$

where, in first approximation, $\bar{\omega} = \omega_0(1 - Gz/h_0)$. Now, for amplitude $\mathcal{E}$ of the electromotive force

$$\mathcal{E}(t) = k_1 \iint_{(S)} \dot{M}_x \, dS,$$

induced in a coil mounted on a cylindrical surface of the specimen (S is the area of the cross section of the specimen), neglecting the change in the amplitude of M_x within the boundaries of the specimen, we find the expression:

$$\mathcal{E} = k \chi_0 H_0 \omega_0 2 \exp(-t/\tau) |I_1(\beta t)| \beta t, \tag{16}$$

where $I_1(\beta t)$ is the Bessel function of first order and $\beta = \gamma G R_0$. The dependence of the amplitude of the signal of free precession $\mathcal{E}/k\chi_0 H_0 \omega_0$ on the time t/τ for different values of $\beta\tau$ is

shown in Fig. 4.

As is seen from the drawing, the effect of the inhomogeneity of the magnetic field does not reduce to a simple decrease in the time constant of damping, as was assumed in [16], but has a more complicated character. As a result of the superposition of the magnetic moments of the different elements of volume of the specimen having different precession velocities, additional maxima arise in the signal. In consideration of the change of amplitude of M_x within the boundaries of the specimen, the character of the damping does not change, however, the signal will not reduce to zero between successive maxima. One can neglect the effect of inhomogeneity for $\beta\tau < 1$. For example, for a specimen of distilled water of diameter 10 cm ($\tau = 3$ sec, $\gamma = 2.67 \times 10^4$ sec^{-1}-oersted^{-1}) one can neglect the effect of inhomogeneity for $G < 2.5 \times 10^{-6}$ oersted-cm^{-1}.

For a small signal-to-noise ratio, when the "beats" are not observed, one can assume approximately that the signal continues to the second zero of the Bessel function $I_1(\beta t)$ [formula (16)]. The durations of the signal of free precession observed in such fashion for $R_0 = 4$ cm agree qualitatively with experimental data of [16] for a gradient from 5×10^{-6} oersted-cm^{-1} to 1.2×10^{-4} oersted-cm^{-1}. As the estimate shows, one can neglect the effect of diffusion [19] on the damping of the signal (for gradients of the field within the limits indicated). In [16] additional maxima of the signal are not

discovered, evidently because of insufficiently high value of the signal-to-noise ratio.

Oscillograms are shown in Fig. 5 of the signal of the free precession for distilled water (τ = 2sec) for an inhomogeneity of the magnetic field of 6.1×10^{-5} oersted-cm^{-1} and 4.2×10^{-5} oersted-cm^{-1}, obtained with the help of the apparatus described in [8]. The inhomogeneity of the earth's magnetic field in the region occupied by the specimen is brought about by a constant magnet with a magnetic moment of 2500 absolute units, placed at distances of 125 and 138 cm from the specimen. For a diameter of the sample equal to 9 cm, the gradient of the magnetic field in its limits can be considered approximately constant. Comparison of the time intervals t_n corresponding to minimum amplitude signal measured by the oscillograms and computed in [16] is given in Table 1. Coincidence of the measured and computed values can be considered good. The initial interval of the precession signal in the oscillograms was not obtained because of the large time required for establishing use of the recording apparatus.

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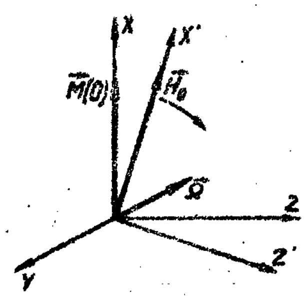


Fig. 1.

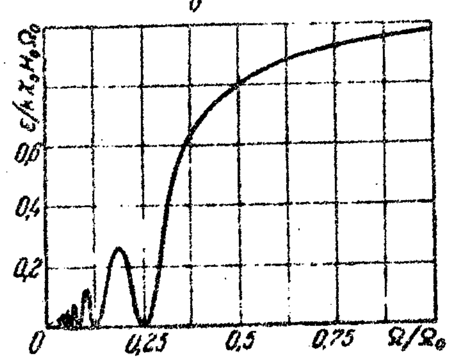


Fig. 2. Dependence of the initial amplitude of the signal of the precession $\mathcal{E}/k\chi_0 H_0 \Omega_0$ on the angular velocity of rotation of the vector of the magnetic field for rotation through 90°.

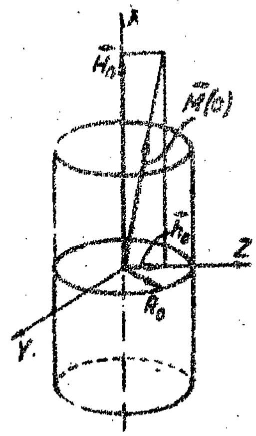


Fig. 3.

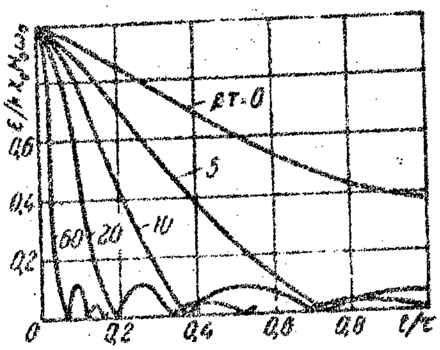
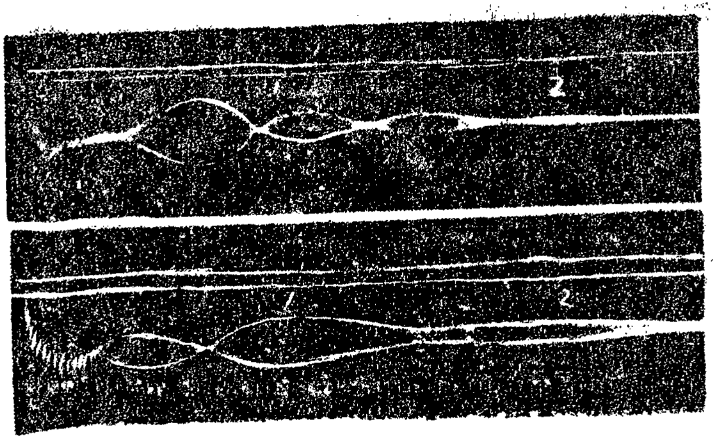


Fig. 4. Dependence of the amplitude of the precession signal $\mathcal{E}/k\chi_0 H\omega_0$ on the time t/τ for different values of $\beta\tau = \gamma G R_0$



a)

б)

Fig. 5. Oscillograms of the signal of free precession for distilled water (τ = 2 sec) for a gradient of the magnetic field inside the specimen of a) G = 6.1 x 10^{-5} oersted-cm^{-1}, b) G = 4.2 x 10^{-5} oersted-cm (scale in sec).

Table 1

1 G (эрстед·см$^{-1}$)	n	2 Измеренное значение t_n (сек)	3 Вычисленное значение t_n (сек)
$4{,}2 \cdot 10^{5}$	1 2	0,72 1,52	0,78 1,40
$6{,}1 \cdot 10^{-5}$	1 2 3 4 5	0,45 0,93 1,35 1,83 2,37	0,52 0,95 1,37 1,80 2,20

(1) G (oersted-cm^{-1}) n (2) Measured value of t_n (sec) (3) Computed value of t_n (sec)

OPTIMUM NONLINEAR SYSTEMS WHICH BRING ABOUT A SEPARATION OF A SIGNAL WITH CONSTANT PARAMETERS FROM NOISE

[This is a translation of an article written by R. L. Stratonovich in Radiofizika (Radiophysics), Vol. II, No. 6, 1959, pages 892-901.]

(Abstract) The a posteriori distribution of the parameters of the of the useful signal which occurs after reception of a signal distorted by a non-Gaussian interference is found by means of the apparatus of Markov processes. Differential equations for optimum filtration are derived; these are modeled by optimum systems.

INTRODUCTION

In practical cases, an interference n(t) exists at the input of a radio receiver designed for reception of a useful signal s(t), in addition to that useful signal. Considering the signal $r(t) = s(t) + n(t)$, we do not assume exact information on the form of the useful signal and on the values of its parameters, since n(t) is an unknown function. However, by accumulating statistical data on the interference n(t) as a random function, we can use these data for the purpose of more

exact determination of the form of the useful signal.

Statistical information on interference and on the useful signal which exists prior to the reception of the entering signal $r(t)$ is known as a priori information. The reception of the signal $r(t)$ materially increases our information about the useful signal. The latter is now described by an a posteriori distribution which has a smaller statistical scatter. To find a posteriori information, it is appropriate to make use of the formula of "inverse probability" [1, 2].

After reception of the oncoming signal $r(t)$, the useful signal remains random, i.e., we cannot say with certainty the exact form of $s(t)$. However, if we select a criterion of optimum behavior, then on the basis of a posteriori information, we can demonstrate the one most preferable form of $s_0(t)$ of the useful signal from all possible such functions. Thus, as the most preferred function $s_0(t)$, we can choose the function which corresponds to the largest a posteriori probability. By choosing another criterion, namely, the criterion of minimum mean square error, we shall have as $s_0(t)$ the mean a posteriori function.

Thus, as the basis of a priori statistical information, we have the representation which the received signal $r(t)$ must satisfy in order that we obtain the most preferred form of the useful signal. It is desirable that the demonstrated representation of $r(t)$ in $s_0(t)$ be obtained automatically by means of a system which we call the optimum filtration system. The construction of such optimum

filtering systems is an extraordinarily important problem for radio communication, automation and other forms of radio technology.

As is well known, in the case of noises and signals which are characterized by Gaussian distributions, the optimum transformation is a linear one and can be computed by the theory of Kolmogorov-Wiener [3,4]. However, many of the signals actually encountered are quite non-Gaussian as, for example, every type of pulse signal and sinusoidal signal with discrete possible amplitude values. In the case of a non-Gaussian signal or noise, the optimum transformation is actually nonlinear. If one seeks the optimum transformation only in the class of linear transformations, then this optimum transformation can be shown to be quite far from the actual optimum transformation. The same can be said about attempts to find an optimum transformation among nonlinear transformations of a more or less broad class [5,6].

We shall therefore seek the absolute optimum transformation, not limiting ourselves to any class of transformations. In the case of a non-Gaussian signal for noise, the search for such a transformation must be carried out on a basis that is new in principle. As investigations have shown, in certain cases, the problem we have stated can be solved with more or less generality. Thus, in the filtration of pulse signals, combined with Gaussian noise, the optimum transformation can be found by use of the theory of correlated random points. In other cases, it is useful to employ the techniques of

Markov processes for the solutions.

When the signal or noise is a Markov process or is the component of a multi-dimensional Markov process, the solution of the problem is made possible by virtue of the fact that it is not difficult to write down an analytic expression for the functional of the probability of the Markov process. In the present work, we shall consider the case in which the noise $n(t)$ is a Markov process, in the general case non-Gaussian, while the signal $s = s(x_1, \ldots, x_m, t)$ is characterized by constant values of the parameters $x_1, \ldots, x_m$, which must then be determined. For fixed values of $x_1, \ldots, x_m$, the signal is a known function of time. Knowing the concrete form of the assumed signal $r(t) = s(x_1, \ldots, x_m, t) + n(t)$ in the time interval from 0 to T, we subject $r(t)$ to analysis in order to extract from it the desired values of the unknown parameters of the signal. Such a treatment (optimum transformation) is performed by analysis of the *a posteriori* distribution of the signal parameters.

1. THE FUNCTIONAL OF THE PROBABILITY OF MARKOV PROCESSES AND THE FORMULA OF INVERSE PROBABILITY

We define the functional of the probability $W[\xi(t)]$ of the random process $\xi(t)$ as an expression depending on $\xi(t)$ which, with accuracy up to a constant factor (which is common for all $\xi(t)$) characterizes the probability of the given form of $\xi(t)$ of this process. The functional of the probability can be obtained by consideration of the joint distribution of the random quantities $\xi(t_1), \ldots, \xi(t_n)$

$(t_1 < \ldots < t_N)$ and the compression of the selected points $t_1, \ldots, t_N$ so that $|t_{j+1} - t_j| \to 0$.

We consider examples of functionals of the probability. Let $\xi(t)$ be a delta correlation ("white") normal noise with zero mean value and correlation function

$$\overline{\xi(t)\,\xi(t')} = K\,\delta(t - t'). \tag{1}$$

Carrying out a partition of the time axis by the points $t_1, \ldots, t_N (t_{j+1} - t_j = \Delta)$, we consider the average valuesover the interval

$$\xi_j = \frac{1}{\Delta} \int_{t_j - \Delta}^{t_j} \xi(t)\, dt. \tag{2}$$

As a consequence of (1), these values are independent of one another, and have the dispersion

$$\overline{\xi_j^2} = K/\Delta. \tag{3}$$

Inasmuch as (2) are Gaussian random quantities, they are described by the distribution density

$$w(\xi_j) = \left(2\pi \frac{K}{\Delta}\right)^{-1/2} \exp\left\{-\frac{\xi_j^2}{2K}\Delta\right\}.$$

Making use of the independence condition, we obtain the joint distribution in the form of a product

$$w(\xi_1, \ldots, \xi_N) = w(\xi_1) \ldots w(\xi_N) = \left(2\pi \frac{K}{\Delta}\right)^{-N/2} \exp\left\{-\frac{1}{2K} \sum_j \xi_j^2 \Delta\right\}.$$

The normalized constant factor does not add any additional information; therefore it can be omitted. Then this functional is

written in the form
$$W[\xi(t)] = \exp\left\{-\frac{1}{2K}\int_0^T \xi^2(t)\,dt\right\}, \tag{5}$$
where T is the length of time interval selected. The integral of this must be taken in the sense of the sum which appears in (4).

Let us consider the noise n(t), which is represented itself by a Markov process. Let it satisfy the equation
$$\dot{n} - f(n) = \xi(t), \tag{6}$$
where $\xi(t)$ is the delta-correlation process described above, and f(n) is a known function. In the general case, for a nonlinear function, f(n) of such a process is non-Gaussian. The probability distribution density W(n) satisfies the Fokker-Planck equation
$$\dot{w}(n) = -\frac{\partial}{\partial n}[f(n)\,w(n)] + \frac{K}{2}\frac{\partial^2 w(n)}{\partial n^2}. \tag{7}$$

It follows from the latter that for small intervals Δ, the probability of transition $W(n_j|n_{j-1})$, which corresponds to the initial condition $n(t_{j-1}) = n_{j-1}$, i.e., the condition $w(n) = \delta(n - n_{j-1})$ for $t = t_{j-1}$, is determined by the expression
$$w(n_j|n_{j-1}) = \left(2\pi\frac{K}{\Delta}\right)^{-1/2}\exp\left\{-\frac{[n_j - n_{j-1} - f(n_{j-1})\Delta]^2}{2K}\Delta\right\}. \tag{8}$$

It follows from the definition of Markov processes that the probability density of a multi-dimensional distribution can be written in the form of a product of transition probabilities:
$$w(n_0, n_1, \ldots, n_N) = w(n_N|n_{N-1})\ldots w(n_1|n_0)\,w(n_0). \tag{9}$$
Substituting (8) in this relation, we obtain

$$w(n_0, n_1, \ldots, n_N) = \text{const}\ \exp\left\{-\frac{1}{2K}\sum_{j=1}^{N}\left[\frac{n_j - n_{j-1}}{\Delta} - f(n_{j-1})\right]^2 \Delta\right\} w(n_0). \tag{10}$$

Squaring $(n_j - n_{j-1})\Delta^{-1} - f(n_{j-1})$, we transform the term $f(n_{j-1})(n_j - n_{j-1})$ to another form. For this purpose, we use several formulas, the first of which is self-evident:

$$f(n_{j-1})(n_j - n_{j-1}) = f\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1}) - \frac{1}{2} f'\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1})^2 + O\,[(n_j - n_{j-1})^3]. \tag{11}$$

For the derivation of the second formula, we take note of the fact that the sum $\sum_\tau (n_j - n_{j-1})^2$, which contains $\tau\Delta^{-1}$ terms, has the mean value

$$\tau\Delta^{-1}\,\overline{(n_j - n_{j-1})^2} = K\tau$$

and the dispersion

$$D\sum(n_j - n_{j-1})^2 = \tau\Delta^{-1} D\,(n_j - n_{j-1})^2 = \tau\Delta^{-1}\,2\,(K\Delta)^2 = 2K^2\tau\Delta.$$

From this follows the equality

$$\sum_{j=q+1}^{q+\tau/\Delta}(n_j - n_{j-1})^2 = K\tau + O\,(\Delta^{1/2}). \tag{12}$$

Dividing the interval [O, T] into sub-intervals of length τ:

$$\frac{1}{f'} \gg \tau \gg \Delta$$

and applying (12), we obtain the formula

$$\sum_j f'\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1})^2 = \sum_l f'(n_l)\,K\tau + O\,(\Delta^{1/2}) + O\,(\tau) \tag{13}$$

$$(n_l = n\,(l\tau)).$$

As a consequence of (11), (13), Eq. (10) can be transformed to the expression

$$w(n_0, n_1, \ldots, n_N) = \text{const}\ \exp\left\{-\frac{1}{2K}\sum_{j=1}^{N}\left[\left(\frac{n_j - n_{j-1}}{\Delta}\right)^2 + \right.\right.$$

$$\left. + f^2(n_{j-1}) - 2f\left(\frac{n_j + n_{j-1}}{2}\right)\left(\frac{n_j - n_{j-1}}{\Delta}\right)\right]\Delta - \tag{14}$$

$$\left. - \frac{1}{2}\sum_{i=1}^{T/\tau} f'(n_i)\,\tau + O(\Delta^{1/2}) + O(\tau)\right\}.$$

In the following, we shall assume that the function f does not depend explicitly on the time. We introduce the potential function

$$U(n) = \int f(n)\, dn. \tag{15}$$

Subtracting the equalities

$$U(n_j) = U\left(\frac{n_j + n_{j-1}}{2}\right) + \frac{1}{2}U'\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1}) +$$

$$+ \frac{1}{8}U''\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1})^2 + O\left((n_j - n_{j-1})^3\right);$$

$$U(n_{j-1}) = U\left(\frac{n_j + n_{j-1}}{2}\right) - \frac{1}{2}U'\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1}) +$$

$$+ \frac{1}{8}U''\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1})^2 + O\left((n_j - n_{j-1})^3\right) \tag{16}$$

from one another, we obtain

$$f\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1}) = U(n_j) - U(n_{j-1}) + O\left((n_j - n_{j-1})^3\right) =$$

$$= U(n_j) - U(n_{j-1}) + O(\Delta^{3/2}). \tag{17}$$

As a consequence of this, part of the terms in (14) are cancelled:

$$\sum_{j=1}^{N} f\left(\frac{n_j + n_{j-1}}{2}\right)(n_j - n_{j-1}) = U(n_T) - U(n_0) + O(\Delta^{1/2}) \tag{18}$$

$$(n_T = n(T); \quad n_0 = n(0)).$$

If we choose the stationary distribution

$$w_0(n_0) = \text{const}\ \exp\left\{-\frac{2}{K}U(n_0)\right\}, \tag{19}$$

as the initial distribution w_0 we then have from (14):

$$\begin{aligned} w(n_0, n_1, \ldots, n_T) = \text{const}\ \exp\Bigg\{ & -\frac{1}{K}[U(n_T) + U(n_0)] - \\ & -\frac{1}{2K}\sum_{j=1}^{N}\left[\left(\frac{n_j - n_{j-1}}{\Delta}\right)^2 + f^2(n_j)\right]\Delta - \\ & -\frac{1}{2}\sum_{l=1}^{T/\tau} f'(n_l)\tau + O(\Delta^{1/2}) + O(\tau)\Bigg\}. \end{aligned} \tag{20}$$

Neglecting the constant factor, and decreasing Δ and τ we obtain the probability functional

$$\begin{aligned} W[n(t)] = \exp\Bigg\{ & -\frac{1}{K}[U(n_T) + U(n_0)] - \\ & -\frac{1}{2K}\int_0^T [\dot{n}^2 + f^2(n)]\,dt - \frac{1}{2}\int_0^T f'(n)\,dt\Bigg\}. \end{aligned} \tag{21}$$

Here $\int n^2 dt$ is understood in the sense of the sum $\sum_j \left(\frac{n_j - n_{j-1}}{\Delta}\right)^2 \Delta$; the other integrals have the usual meaning.

Knowing the form of the received signal and the form of the functional of the probability, we can compute the a posteriori distribution of the parameters of the useful signal, making use of the principle of inverse probability.

Let the parameters $x_1, \ldots, x_m$ be described by the a priori distribution $w_{pr}(x_1, \ldots, x_m)$. For fixed values of the parameters $x_1, \ldots, x_m$ the effective signal $r(t) = s(x_1, \ldots, x_m, t) + n(t)$ would have the a priori distribution

$$W'[r \mid x_1, \ldots, x_m] = W'[n]_{n=r-s} \equiv W'_n[r-s]. \tag{22}$$

Therefore the joint distribution for $x_1, \ldots, x_m$ and $r(t)$ has the form

$$W'[x_1, \ldots, x_m, r] = w_{pr}(x_1, \ldots, x_m)\; W'_n[r-s]. \tag{23}$$

By the definition of conditional probabilities, it is also possible to write them in the form

$$W'[x_1, \ldots, x_m, r] = W'[r]\; w(x_1, \ldots, x_m \mid r). \tag{24}$$

where $w(x_1, \ldots, x_m | r) = w_{ps}(x_1, \ldots, x_m)$ is the conditional or a posteriori distribution of the parameters. Comparing (23) and (24), we find

$$w_{ps}(x_1, \ldots, x_m) = Cw_{pr}(x_1, \ldots, x_m)\ W_n\,[r(t) - s(x_1, \ldots, x_m, t)], \tag{25}$$

where the constant C does not depend on $x_1, \ldots, x_m$.

With the help of the a posteriori distribution density thus obtained, one can determine the preferred values of the parameters. If we take as the latter the a posteriori mean, then we shall have:

$$(x_k)_0 = \frac{\int \ldots \int x_k\, w_{pr}(x_1, \ldots, x_m)\ W_n\,[r(t) - s(x_1, \ldots, x_m, t)]\, dx_1 \ldots dx_m}{\int \ldots \int w_{kr}(x_1, \ldots, x_m)\ W_n\,[r(t) - s(x_1, \ldots, x_m, t)]\, dx_1 \ldots dx_m} \tag{26}$$

$$(k = 1, \ldots, m).$$

In the same way we found the optimum transformation, which is non-linear in the general case and which is adjusted in the best fashion for obtaining the unknown parameters. A solution for Gaussian noises was obtained in [7] in analogous form.

However, the solution of the problem that we have written

down has two shortcomings. In the first place, the optimum transformation that was obtained contains integral operations which can be shown to be complicated for the practical realization of optimum systems. It is easier to carry out operations corresponding to differential equations. In the second place, the method of solution that we have shown does not permit generalization to the case of parameters which change with time. Therefore, we shall consider another solution of the problem.

2. THE DIFFERENTIAL EQUATIONS OF OPTIMUM FILTRATION

In order to obtain the equations of optimum filtration in differential form, let us study the change in the a posteriori distribution upon increase in the time of observation. Let the length of the interval of observation T be increased by dT. The change in the a posteriori density in this case can be found by using Eq. (25). Taking the logarithm of both sides of the equation and differentiating with respect to T, we obtain

$$\frac{d \ln w_{ps}}{dT} = \frac{d}{dT} \ln W_n [r - s] + \frac{d \ln C}{dT}. \tag{27}$$

Here we have taken it into account that w_{pr} does not depend on T, inasmuch as we shall consider only the constant parameters of the signal. Another paper will be devoted to a generalization of this theory to the case of variable parameters of the signal, in which w_{pr} changes with time.

If we introduce the notation

$$F(r - s) = \frac{d}{dT} \ln W_n [r - s], \tag{28}$$

then we can write (27) in the form:

$$\dot{w}_{ps} = [F(r - s) - \lambda]\, w_{ps} \tag{29}$$

(we shall indicate differentiation with respect to T and also with respect to t by means of the dot). The constant here $\lambda = -d \ln C/dT$ is connected with the normalization constant. It can be determined by employing the fact that the integral $\int \dots \int \dot{w}_{ps} dx_1, \dots, dx_m$ vanishes by virtue of the normalization condition. Therefore, integration of (29) over $x_1, \dots, x_m$ gives:

$$\lambda = \int \dots \int F(r - s)\, w_{ps}\, dx_1, \dots, dx_m = \overline{F(r - s)}.$$

The resultant differential equation $\dot{w}_{ps} = (F - \overline{F})w_{ps}$ is thus a nonlinear differential equation. It would have been advantageous to use the system which models this equation for purposes of separation of the parameters of the signal. However, direct modeling of Eq. (29) is difficult for the reason that, for continuous possible values of the parameters $x_1, \dots, x_m$, it corresponds to an unbounded number of degrees of freedom. It is more suitable to consider the evolution not of the a posteriori density itself but certain numerical characteristics of it, and to model the equations which describe their change with respect to time.

We consider the case in which there is one unknown parameter x of the signal, taking on continuous values. In place of the distribution density $w_{ps}(x)$, we can consider its replacement by the numerical characteristics $a_1, a_2, a_3, \dots,$ which represents the coefficients of the expansion of its logarithm in a Taylor series. The

corresponding connection formula has the form

$$w_{ps}(x) = \exp\left[c_0 + \frac{a_2}{2}(x - a_1)^2 + \frac{a_3}{3!}(x - a_1)^3 + \ldots\right]$$
$$(a_2 \equiv -a < 0). \tag{30}$$

The quantity a_1 is the most probable value of the parameter. It is useful to choose it as the most preferable value of x_0. The coefficients we have introduced change during the course of observation (they depend on T). Substituting (30) in (29), we obtain:

$$\dot{a}_2 \frac{(x - a_1)^2}{2} + \dot{a}_3 \frac{(x - a_1)^3}{3!} + \ldots - a_2\dot{a}_1 (x - a_1) -$$
$$- a_3\dot{a}_1 \frac{(x - a_1)^2}{2} + \ldots = F(r - s) - \lambda - \dot{c}_0.$$

Taking the derivative with respect to x of both sides of the equation at the point $x = a_1$, we shall have a system of equations which define the evolution of the coefficients $a_1, a_2, a_3 \ldots$ during the course of observation

$$\begin{aligned} -a_2\dot{a}_1 &= [\partial F/\partial x]_{x=a_1}; \\ \dot{a}_2 - a_3\dot{a}_1 &= [\partial^2 F/\partial x^2]_{x=a_1}; \\ &\cdots\cdots\cdots\cdots \\ \dot{a}_n - a_{n+1}\dot{a}_1 &= [\partial^n F/\partial x^n]_{x=a_1}; \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{31}$$

As initial values of $a_1, a_2, a_3, \ldots$ at $T = 0$, we should take those values which correspond to the a priori distribution density $w_{pr}(x)$. In the case in which the a priori information on the parameters of the useful signal are absent, $w_{pr}(x)$ is a constant. In this case one should choose the zero initial conditions:

$$a_2(0) = 0; \quad a_3(0) = 0. \tag{32}$$

The value of $a_1(0) = x$ is determined from the condition of the

maximum of the function F(r - s) at the particular moment:

$$\partial F\,|r\,(0) - s\,(x, 0)|/\partial x = 0. \qquad (33)$$

Under the conditions (32) for the momentsof time which are not too far from the initial moment, the term $a_{n+1}\dot{a}_1$ in (31) is much smaller than the term $a_{m+1}\ \dot{a}_1$. One can use this to break the chain of equations (31), leaving only n equations relative to the variables $a_1, a_2, \ldots, a_n$, and setting $a_{n+1} = 0$. The larger the value of n, the smaller the error we make. If we have two equations, then

$$\partial \dot{x}_0 = |\partial F/\partial x|_{x=x_0}; \qquad \dot{a} = -|\partial^2 F/\partial x^2|_{x=x_0}$$
$$(x_0 = a_1; \qquad a = -a_2) \qquad (34)$$

for the initial conditions $a(0) = 0$, $\partial F/\partial x = 0$ for $x = x_0(0)$.

The system which models these equations will give at the output $x_0(T)$ as the value of the parameter which corresponds to the maximum a posteriori probability, and the quantity a, by means of which the error of the resultant value is expressed according to the formula $\Delta x = 1/\sqrt{a}$ (Δ x is the order of the difference of x_0 from the real value of x). As is seen from (34), $x_0(T)$ initially depends strongly on T; thereafter this dependence weakens, corresponding to the fact that new information obtained from r(t) makes up a small fraction of all the received information. The error Δ x decreases without limit during the passage of the time of observation.

Equations (34) refer to the case in which there is a single unknown parameter of the signal. In the case of many parameters $x_1, \ldots, x_m$, the following formulas are obtained by a similar

method in the same approximation:

$$\sum_{\beta=1}^{m} a_{\alpha\beta}\dot{x}_\beta = \partial F/\partial x_\alpha ;$$

$$\dot{a}_{\alpha\beta} = -\,\partial^2 F/\partial x_\alpha \partial x_\beta \qquad (\alpha, \beta = 1, \ldots, m), \tag{35}$$

where $x_1, \ldots, x_m$ are the preferred values of the parameters.

3. EXAMPLE. DETERMINATION OF THE DEVIATION IN NON-GAUSSIAN NOISE

We shall take a concrete case of the differential equation of optimum filtration worked out above for the situation in which the useful signal represents an unknown constant quantity. During the passage of time from 0 to T, the total signal $r(t) = s + n(t)$ is received, where $n(t)$ is a Markovian random process, non-Gaussian in the general case. Let us satisfy Eq. (6) and, consequently, be described by the functional of probability (21).

Making use of Eqs. (38) and (21), we have in the given case:

$$F(r-s) = -\frac{1}{2K}\,|\dot{r} - f(r-s)|^2 - \frac{1}{2}f'(r-s). \tag{36}$$

Differentiating the resultant expression, we find the corresponding equation of optimum filtration (34), which takes the form:

$$Ka\dot{s}_0 = -\,[\dot{r} - f(r-s_0)]\,f'(r-s_0) + \frac{K}{2}f''(r-s_0);$$

$$K\dot{a} = -\,[\dot{r} - f(r-s_0)]\,f''(r-s_0) + f'^2(r-s_0) - \frac{K}{2}f'''(r-s_0) \tag{37}$$

$$[f'(n) = \partial f(n)/\partial n].$$

In the absence of a priori information on the signal, the initial value of $a(0)$ is equal to zero, while $s_0(0)$ is determined from the condition of the minimum of the function $F(r - s_0)$. Differentiating it, we

obtain the result that it achieves its extremum value for

$$|\dot{r}(0) - f(r(0) - s_0(0))| f'(\dot{r}(0) - s_0(0)) + \\ + \frac{K}{2} f''(r(0) - s_0(0)) = 0. \tag{38}$$

Further contributions will be realized for the case in which one can neglect the term $-1/2f'(r - s)$ in (36) in comparison with the first term. In this case the condition for the initial value of $s_0(0)$, for which the function (36) takes on its maximum value has the form:

$$f|r(0) - s_0(0)| = \dot{r}(0). \tag{39}$$

Equations (37) with the given initial conditions determine the evolution of the preferred value of the signal (s_0T) during the course of observation completely and unambiguously. It is true, the value of the derivative at the initial instant is not entirely explicit from the first equation. In order to find it, one must clear up the indeterminacy. As is seen from the second equation (37), we have for small T:

$$Ka \simeq f'^2 |r(0) - s_0(0)| T.$$

At the same time, for small T,

$$\dot{r}(T) - f|r(T) - s_0(T)| \simeq \ddot{r}(0) T - f'|r(0) - \\ - s_0(0)| |\dot{r}(0) - \dot{s}_0(0)| T.$$

Substituting these relations in the first equation of (37), and dividing by T, we obtain:

$$f'^2(r - s_0)\dot{s}_0 = -|\ddot{r} - f'(r - s_0)(\dot{r} - \dot{s}_0)| f'(r - s_0).$$

We then determine the value of the derivative $\dot{s}_0$ at the initial instant:

$$\dot{s}_0(0) = \frac{1}{2}\left[\dot{r} - \frac{\ddot{r}}{f'(r - s_0)}\right]_{t=0}. \tag{40}$$

The value of a increases with increase in T. Therefore, as is seen in the first equation of (37), the value of the derivative $\dot{s}_0(T)$ in what follows decreases, and the a posteriori signal $s_0(T)$ approaches a constant value, which coincides with the actual value of the useful field. The error $\Delta s \sim 1/\sqrt{a}$ in this case decreases to zero.

The resultant equations (37) are nonlinear. In the special case in which f(n) is a linear function, they reduce to linear equations. Setting $f(n) = \beta - n$, we find:

$$Ka\dot{s}_0 = \beta\,[\dot{r} + \beta r - \beta s_0]; \quad K\dot{a} = \beta^2. \tag{41}$$

The latter equation has the solution

$$Ka = \beta^2 T. \tag{42}$$

Substitution of it in the first term gives the equation

$$T\dot{s}_0 + s_0 = r + \dot{r}/\beta$$

(with the initial condition (38): $s(0) = r(0) + \dot{r}(0)/\beta$), whose solution is written in the form:

$$s_0(T) = \frac{1}{T}\int_0^T \left[r(t) + \frac{\dot{r}(t)}{\beta}\right] dt. \tag{43}$$

This result could have been obtained by another way, directly using Eq. (26). Indeed, for a linear function f(n), the noise n(t) is Gaussian, and the solution of the problem is materially simplified. For non-Gaussian noise, other more appropriate forms of solution are

absent and it is necessary to make use of Eqs. (37).

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ON THE EXCITATION OF DIELECTRIC WAVEGUIDES

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(Abstract) The problem is considered of the excitation at the open end of a semi-infinite waveguide of surface electromagnetic waves on a dielectric slab placed on an ideally conducting plane and continued inside the waveguide. A rigorous solution was obtained by the method of integral equations in the case in which the system as a whole is screened by an ideally conducting plane placed at a certain distance from the dielectric plate, parallel to the latter. Formulas are obtained for the fields of the reflected and transformed waves; the moduli of the coefficients of transformation of the traveling wave in the propagated wave are represented in closed form and are determined only by the wave numbers of the propagated waves. The expressions for the moduli of

coefficients of reflection and transformation of the fundamental waves can be used for approximate calculation of the corresponding quantities in the unscreened system if one assumes that the screening plane is placed at a sufficient distance from the dielectric plate.

Excitation of surface waves in dielectric waveguides and antennas [1] usually takes place at the open end of the waveguide. Experimental studies [2] have shown that in this case a rather significant fraction of the power of the incident wave is not transformed into the energy of the surface wave, but is lost in radiation directly from the end of the exciting waveguide. This radiation has a known effect on the directivity diagram of dielectric antennas, and also lowers the resultant coefficient of transformation of dielectric wave guides.

In connection with this fact, it is of interest to estimate quantitatively the effectiveness of such a method of excitation. A similar problem on the excitation of surface waves on the impedance plane was considered in [3]. For a dielectric waveguide, the calculation can be completed in principle, at least in the simplest cases by the same method which was used in [3], but this is combined with great difficulties of factorization of the kernel of the integral equation.

In order to avoid these difficulties and at the same time to

make possible to calculate the amplitude of the excited surface wave with sufficient accuracy (practically with as much accuracy as desired), let us consider the problem of excitation not of an open, but of a screened dielectric waveguide, in which we limit ourselves to the simpler case of the excitation of two-dimensional H waves on a dielectric slab lying on an ideally conducting plane. Consideration of the screened system materially simplifies the problem of factorization of the kernel of the integral equation.

1. STATEMENT OF THE PROBLEM. DERIVATION AND SOLUTION OF THE INTEGRAL EQUATION

Let us consider the system shown in Fig. 1. It represents a dielectric slab of thickness d with parameters ε_1, μ_1, lying in an ideally conducting plane $x = 0$. The exciting waveguide is formed by the plane $x = 0$ and the ideally conducting half-plane $x = d$, $z < 0$, set directly on the surface of the dielectric slab. At a distance D from the plane $x = 0$, there is placed an ideally conducting screen sheet $(x = D > d)$.

Limiting ourselves to a consideration of two-dimensional H waves ($H_y \neq 0$, $E_y = 0$), we represent the diffraction current $I_z = I(z)$, excited by the incident wave of a definite type on the half-plane $x = d$, $z < 0$, by the Fourier integral*

$$I(z) = \int_{-\infty}^{\infty} F(h) e^{-ihz} dh . \tag{1.1}$$

*The time dependence $e^{i\omega t}$ is everywhere omitted.

The component $H_y \equiv H$ of the magnetic field of this current in the dielectric (I) and on top of the dielectric (II), respectively, will be described by the equations:

$$H_I' = \int_{-\infty}^{\infty} C_1(h) \cos(\varkappa_1 x)\, e^{-ihz}\, dh ;$$
$$(1.2)$$
$$H_{II}' = \int_{-\infty}^{\infty} C_2(h) \cos[\varkappa_2 (x-D)]\, e^{-ihz}\, dh ,$$

where

$$C_1(h) = \frac{\varkappa_2 \sin[\varkappa_2(x-D)]}{\varepsilon^{-1}\varkappa_1 \sin(\varkappa_1 d)\cos[\varkappa_2(d-D)] - \varkappa_2 \sin[\varkappa_2(d-D)]\cos(\varkappa_1 d)} F(h); \quad (1.3)$$

$$C_2(h) = \frac{\varepsilon^{-1}\varkappa_1 \sin(\varkappa_1 d)}{\varepsilon^{-1}\varkappa_1 \sin(\varkappa_1 d)\cos[\varkappa_2(d-D)] - \varkappa_2 \sin[\varkappa_2(d-D)]\cos(\varkappa_1 d)} F(h); \quad (1.4)$$

$$\varkappa_1 = \sqrt{k_1^2 - h^2}; \quad \varkappa_2 = \sqrt{k_2^2 - h^2} \quad \left(k_{1,2} = \omega\sqrt{\varepsilon_{1,2}\mu_{1,2}}, \quad \varepsilon = \varepsilon_1/\varepsilon_2\right). \quad (1.5)$$

Using (1.2) and satisfying the boundary conditions on the plane p = d, we obtain the following set of integral equations for the function F(h):

$$\int_{-\infty}^{\infty} F(h)\, L(h)\, e^{-ihz}\, dh = 0 \quad (z < 0);$$
$$(1.6)$$
$$\int_{-\infty}^{\infty} F(h)\, e^{-ihz}\, dh = -I_n e^{-ih_n^{(1)} z} \quad (z > 0),$$

where I_n is the amplitude of the current on the plane x = d, brought about by the field of the incident wave of number n with propagation constant $h_n^{(1)}$,

$$L(h) = \varkappa_2 \varphi^{(1)}(h)\, \varphi^{(2)}(h)/\varphi^{(3)}(h); \quad (1.7)$$

$$\varphi^{(1)}(h) = 1 - e^{-2i\varkappa_1 d};\qquad(1.8)$$

$$\varphi^{(2)}(h) = 1 - e^{-2i\varkappa_2(D-d)};\qquad(1.9)$$

$$\varphi^{(3)}(h) = \varepsilon^{-1}\left[1 - e^{-2i\varkappa_1 d}\right]\left[1 + e^{-2i\varkappa_2(D-d)}\right] + \varkappa_1^{-1}\varkappa_2\left[1 + e^{-2i\varkappa_1 d}\right]\left[1 - e^{-2i\varkappa_2(D-d)}\right].\qquad(1.10)$$

The set of equations (1.6) is of the same type as those considered in [3, 4]. For their solution, it is necessary to factorize the kernel L(h), i.e., to find the auxiliary functions $L_1(h)$ and $L_2(h)$ which bring about the expansion of the function

$$L(h) = L_1(h)\,L_2(h)\qquad(1.11)$$

in terms of factors holomorphic in the upper and lower half-planes of h, respectively. Then the solution of Eq. (1.6) will be the function

$$F(h) = \frac{I_n}{2\pi i}\,\frac{L_2\left(h_n^{(1)}\right)}{L_2(h)}\,\frac{1}{h - h_n^{(1)}}.\qquad(1.12)$$

2. FREE WAVES

The equations $\varphi^{(p)}(h) = 0$ (p = 1, 2, 3), which are equivalent, respectively, to the equations

$$\sin(\varkappa_1 d) = 0;\qquad(2.1)$$

$$\sin\left[\varkappa_2(D-d)\right] = 0;\qquad(2.2)$$

$$\varepsilon^{-1}\varkappa_1\,\mathrm{tg}(\varkappa_1 d) = -\varkappa_2\mathrm{tg}\left[\varkappa_2(D-d)\right],\qquad(2.3)$$

determine, together with (1.5), the propagation constants of the particular types of waves $h_m^{(p)}$ (p = 1, 2, 3) in regular waveguides 1, 2, 3

(see Fig. 1).

The solutions of the first two equations (2.1), (2.2) are obvious:

$$\varkappa_{1m} = m\,\pi/d\,; \quad h_m^{(1)} = \sqrt{k_1^2 - \varkappa_{1m}^2} \quad (m = 0,\ 1,\ 2,\ \ldots)\,; \tag{2.4}$$

$$\varkappa_{2m} = m\,\pi/(D-d)\,; \quad h_m^{(2)} = \sqrt{k_2^2 - \varkappa_{2m}^2} \quad (m = 0,\ 1,\ 2,\ \ldots)\,. \tag{2.5}$$

The roots $\varkappa_{1m}^{(3)}$, $\varkappa_{2m}^{(3)}$ and $h_m^{(3)}$ of Eqs. (2.3) and (1.5) can be found graphically. It is important to note that besides the infinite number of roots $h_m^{(3)}$, corresponding to real eigenvalues $\varkappa_{2m}^{(3)}$, these equations always have at least one, and for sufficiently thick dielectric plates and large ε, several roots, which correspond to imaginary eigenvalues $\varkappa_{2m}^{(3)} = i\,\tilde{\varkappa}_{2m}^{(3)}$. These roots are contained in the range $k_2 \leq h \leq k_1$, and give the wave numbers of "slow" waves, the phase velocity of which is less than the velocity of plane transverse waves in a medium with parameters ε_2, μ_2.

The dependence of the dimensionless constants of propagation $\Gamma_m^{(3)} = h_m^{(3)} d$ of the first types of waves (m = 0, 1, 2, 3) on $\xi = (D - d)/d$ are given in Figs. 2, 3 for the following values of the parameters: $\mu_1/\mu_2 = \mu = 1$; $\varepsilon = 2.\varepsilon$; $k_2 d = 1$. The values $\xi = \xi_{crm}$ (m = 1,2,...) are critical for the corresponding types of waves: for $\xi < \xi_{crm}$, the propagation constant $h_m^{(3)}$ becomes imaginary, and the wave of the given type cannot be propagated in the waveguide. For the fundamental wave m = 0 such a critical value of the parameter ξ does not exist.

Setting $\varkappa_2 = i\,\tilde{\varkappa}_2$ in (2.3) and assuming that the size of

D - d to be sufficiently large, we obtain the equation

$$\varepsilon^{-1} \varkappa_1 \operatorname{tg} (\varkappa_1 d) = \bar{\varkappa}_2 , \qquad (2.6)$$

which, together with (1.5) determines the propagation constants of surface waves on the dielectric plate. Equation (2.6) has been studied in sufficient detail in a number of researches [5, 6].

We shall write down the expressions for the components of $H_y \equiv H$ of the field of free waves in regular waveguides 1, 2, 3:

$$\begin{aligned}
H^{(1)}_{\perp m} &= A^{(1)}_m \cos (\varkappa_{1m} x)\, e^{\mp i h^{(1)}_m z} ;\\
H^{(2)}_{\perp m} &= A^{(2)}_m \cos | \varkappa_{2m} (x - D) |\, e^{\mp i h^{(2)}_m z} ;\\
H^{(3)}_{I \perp m} &= A^{(3)}_m \varkappa^{(3)}_{2m} \sin | \varkappa^{(3)}_{2m} (d - D) | \cos \left(\varkappa^{(3)}_{1m} x \right) e^{\mp i h^{(3)}_m z} ;\\
H^{(3)}_{II \perp m} &= A^{(3)}_m \varepsilon^{-1} \varkappa^{(3)}_{1m} \sin \left(\varkappa^{(3)}_{1m} d \right) \cos | \varkappa^{(3)}_{2m} (x - D) |\, e^{\mp i h^{(3)}_m z} .
\end{aligned} \qquad (2.7)$$

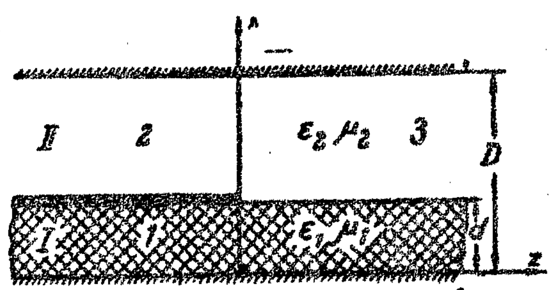

Fig. 1.

The field (2.7) for all waveguides should be so normalized that the partial energy flux (complex) in the direction of the z axis would be equal to

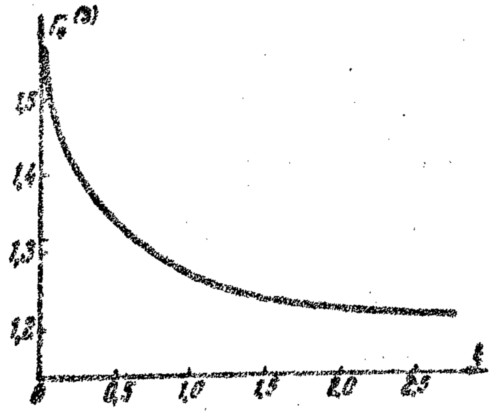

Fig. 2.

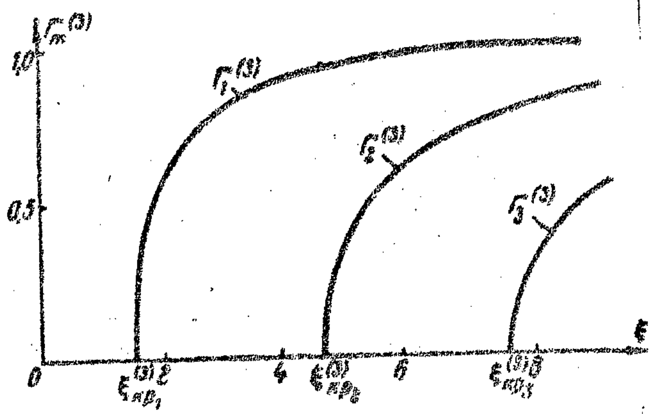

Fig. 3.

$$\widetilde{S}_m^{(p)} = \frac{1}{2} \int_{S^{(p)}} E_x H_y^* \, dx = \frac{h_m^{(p)}}{|h_m^{(p)}|}. \tag{2.8}$$

For the time average of the energy flux of the propagating normalized wave, we have:

$$S_m^{(p)} = \operatorname{Re} \widetilde{S}_m^{(p)} = 1 . \tag{2.9}$$

For the given normalization of the field (2.8), the expressions for the amplitude $A_m^{(p)}$ will be the following:

$$A_m^{(1)} = 2\sqrt{\frac{k_1}{Z_1 h_m^{(2)} d}}; \qquad A_m^{(2)} = 2\sqrt{\frac{k_2}{Z_2 h_m^{(2)}(D-d)}};$$

$$A_m^{(3)} = \frac{2}{\sqrt{h_m^{(3)}}} \Big\langle \frac{Z_1}{k_1} \varkappa_{2m}^{(3)2} \sin^2 \left| \varkappa_{2m}^{(3)} (d-D) \right| \left[d + \frac{1}{2\varkappa_{1m}^{(3)}} \sin\left(2\varkappa_{1m}^{(3)} d\right) \right] +$$

$$+ \frac{Z_2}{k_2} \frac{\varkappa_{1m}^{(3)2}}{\varepsilon^2} \sin^2(\varkappa_{1m}^{(3)} d) \left\{ D - d + \frac{1}{2\varkappa_{2m}^{(3)}} \sin\left|2\varkappa_{2m}^{(3)}(D-d)\right| \right\} \Big\rangle^{-1/2}, \qquad (2.10)$$

where $Z_p = \sqrt{\mu_p/\varepsilon_p}$ (p = 1, 2).

Assuming the incident wave (of the type $H_n^{(1)}$) to be normalized, and making use of the boundary condition for the tangential component of the magnetic field on an ideal conductor, we obtain the following expression for the current I_{zn} on the plane x = d, connected with the field of the incident wave:

$$I_{zn} = (-1)^{n+1}\, 2\sqrt{k_1/Z_1 h_n^{(1)} d}\; e^{-ih_n^{(1)} z}. \qquad (2.11)$$

We then find the explicit expression for the amplitude I_n in (1.6):

$$I_n = (-1)^{n+1}\, 2\sqrt{k_1/Z_1 h_n^{(1)} d}.$$

3. DETERMINATION OF THE AUXILIARY FUNCTIONS

To find the auxiliary functions $L_1(h)$ and $L_2(h)$, we consider separately each of the factors entering into the expression for L(h). We assume that the quantities k_1 and k_2 possess such a small absolute value of the negative imaginary part that in the final results we can set Im k_1 = Im k_2 = 0.

The expansion of $\varkappa_2$ is evidently

$$\varkappa_2 = \sqrt{k_2 - h}\;\sqrt{k_2 + h}. \qquad (3.1)$$

The functions which make possible the expansion

$$\varphi^{(p)}(h) = \varphi_1^{(p)}(h)\,\varphi_2^{(p)}(h) \qquad (p = 1,2), \tag{3.2}$$

were obtained in reference [4]:

$$\varphi_1^{(p)}(h) = N^{(p)}(1-h/k_p)^{1/2}\prod_{m=1}^{\infty}(1-h/h_m^{(p)})\exp\left[-\pi^{-1}d_p\left(\tau_m^{(p)}-\tau_{m-1}^{(p)}\right)h - \right.$$
$$\left. - ik_p d_p M\,\pi^{-1}(\tau^{(p)})\right]; \tag{3.3}$$
$$\varphi_2^{(p)}(h) = \varphi_1^{(p)}(-h),$$

where

$$M(\tau^{(p)}) = (\pi/2 + \tau^{(p)})\cos\tau^{(p)} - \sin\tau^{(p)};$$

$$N^{(p)} = \sqrt{2i\sin(k_p d_p)};\quad k_p\sin\tau^{(p)} = h;\quad k_p\sin\tau_m^{(p)} = h_m^{(p)}\ (m=1,2,\ldots);\ \tau_0 = \pi/2,$$

where $d_1 = d$, $d_2 = D - d$.

The auxiliary functions $\phi_1^{(3)}(h)$ and $\phi_2^{(3)}(h)$ for the expansion

$$\varphi^{(3)}(h) = \varphi_1^{(3)}(h)\,\varphi_2^{(3)}(h) \tag{3.4}$$

can be obtained by well-known formulas [4, 7]. The special characteristic here in comparison with the computation of auxiliary functions $\phi_{1-2}^{(p)}$ (p = 1,2), and also of the functions given in [3] is the presence of two cuts: Im $\varkappa_1 = 0$ and Im $\varkappa_2 = 0$ in each of the half-planes of the complex variable h, while in the limiting case Im k_1 = Im k_2 = 0, all the roots of the characteristic equation (2.3) lie on these cuts (the part of the roots with Re $h_n^{(3)} > k_2$ lie only on the cut Im $\varkappa_1 = 0$). Omitting the rather involved intermediate calculations, we only write out the final result:

$$\varphi_1^{(3)}(h) = N^{(3)}(1-h/k_1)^{-1/4}\exp\left[-\frac{ik_1 d}{\pi}M(\tau_1) - \frac{ik_2(D-d)}{\pi}M(\tau_2)\right]\times$$
$$\times\prod_{m=0}^{\infty}(1-h/h_m^{(3)})\exp\left[-\frac{d}{\pi}\left(\tau_{1m}^{(3)}-\tau_{1m-1}^{(3)}\right)\right]\prod_{m=k+1}^{\infty}\exp\left[-\frac{D-d}{\pi}\left(\tau_{2m}^{(3)}-\tau_{2m-1}^{(3)}\right)h\right]; \tag{3.5}$$
$$\varphi_2^{(3)}(h) = \varphi_1^{(3)}(-h),$$

where

$$N^{(3)} = 2e^{i\pi/4}\left\{\frac{1}{\varepsilon}\sin(k_1 d)\cos[k_2(D-d)] + \frac{1}{\sqrt{\varepsilon\mu}}\cos(k_1 d)\sin[k_2(D-d)]\right\}^{1/2};$$

$$\sin\tau_p = h/k_p; \quad \sin\tau_{pm}^{(3)} = h_m^{(3)}/k_p \quad (p = 1,\ 2),$$

$\mathrm{Re}\, h_m^{(3)} > k_2$ for $m = 0, 1, \ldots, k$ and $\mathrm{Re}\, h_m^{(3)} < k_2$ for $m = k + 1, k + 2, \ldots$. In (3.5), it must be assumed that $\tau_{1,-1}^{(3)} = \pi/2$ under the sign of the first product, and $\tau_{2k}^{(3)} = \pi/2$ under the sign of the second product. All the auxiliary functions $\phi_{1\cdot 2}^{(1,2,3)}(h)$ given above are limited to finite ranges in those half-planes where they are holomorphic [7].

The auxiliary functions $L_1(h)$, $L_2(h)$ can evidently be written down in terms of the functions $\phi_{1\cdot 2}^{(1,2,3)}(h)$ in the following way:

$$L_1(h) = \sqrt{k_2 - h}\ \varphi_1^{(1)}(h)\ \varphi_1^{(2)}(h)/\varphi_1^{(3)}(h);$$
$$L_2(h) = \sqrt{k_2 + h}\ \varphi_2^{(1)}(h)\ \varphi_2^{(2)}(h)/\varphi_2^{(3)}(h). \qquad (3.6)$$

We shall develop expressions for the squares of the moduli of the auxiliary functions $\phi_{1\cdot 2}^{(1,2,3)}(h)$ which are necessary in what follows for real values of h.

Assuming that in each of the waveguides 1, 2, 3, only $i^{(p)} + 1$ waves ($p = 1,2,3$) can be propagated, so that the propagation constants $h_1^{(p)}$, $h_2^{(p)}$, $h_{i(p)}^{(p)}$ are real numbers,* while $h_m^{(p)} = -i\hbar_m^{(p)}$

*We recall that the fundamental types of waves in these waveguides have zero critical frequencies.

($\mathrm{Im}\,\hbar_m^{(p)} = 0$ for $m = i^{(p)} + 1$, $i^{(p)} + 2, \ldots$) and, making use of the

expression

$$|\varphi_1^{(p)}|^2\,|\varphi_2^{(p)}|^2 = |\varphi^{(p)}|^2, \tag{3.6}$$

we find that

$$|\varphi_1^{(p)}|^2 = \left|2\sin\varkappa_p d_p\, e^{-i\varkappa_p d_p + \nu^{(p)} d_p + h d_p}\left(\frac{1-h/k_p}{1+h/k_p}\right)^{1/2} \prod_{m=1}^{i^{(p)}} \frac{1-h/h_m^{(p)}}{1+h/h_m^{(p)}}\right| \ (p=1,2);$$

(3.7)

$$|\varphi_1^{(3)}|^2 = \left|\varphi^{(3)}(h)\, e^{\nu^{(1)} d + \nu^{(2)}(D-d) + hD}\left(\frac{1+h/k_1}{1-h/k_1}\right)^{1/2} \prod_{m\ 0}^{i^{(3)}} \frac{1-h/h_m^{(3)}}{1+h/h_m^{(3)}}\right|;$$

$$|\varphi_2^{(p)}(h)|^2 = |\varphi_1^{(p)}(-h)|^2 \qquad (p=1,\ 2,\ 3),$$

where

$$\nu^{(p)} = \begin{cases} -\tilde{\varkappa}_p & (h \geqslant k_p) \\ 0 & (|h| \leqslant k_p) \\ \tilde{\varkappa}_p & (h \leqslant -k_p) \end{cases}; \tag{3.8}$$

$$\tilde{\varkappa}_p = \sqrt{h^2 - k_p^2} \qquad (p=1,\ 2).$$

In (3.7), it must be assumed that $\varkappa_p = -i\,\tilde{\varkappa}_p$ for $|h| > k_p$.

4. EXPRESSIONS FOR THE FIELDS. TRANSFORMATION COEFFICIENTS

Substituting the solution (1.12) for the function F(h) in (1.2), it is not difficult to establish the fact that the only singular points of the integrands in (1.2) are simple poles in the roots of the characteristic equations (2.1)-(2.3). Applying the theorem of Cauchy, we obtain for the fields in the waveguides 1, 2, 3,

$$H^{(1)} = \sum_{l=0}^{\infty} T_{nl}^{(11)} H_{-l}^{(1)} + H_{+n}^{(1)} ; \tag{4.1}$$

$$H^{(2)} = \sum_{l=0}^{\infty} T_{nl}^{(12)} H_{-l}^{(2)} ; \tag{4.2}$$

$$H^{(3)} = \begin{cases} H_{\mathrm{I}}^{(3)} = \sum\limits_{l=0}^{\infty} T_{nl}^{(13)} H_{\mathrm{I}+l}^{(3)} & (0 \leqslant x \leqslant d) \\ H_{\mathrm{II}}^{(3)} = \sum\limits_{l=0}^{\infty} T_{nl}^{(13)} H_{\mathrm{II}-l}^{(3)} & (d \leqslant x \leqslant D) \end{cases}, \tag{4.3}$$

where
$$T_{nl}^{(11)} = -\frac{iI_n}{A_l^{(1)} B_l^{(1)}} \frac{L_2(h_n^{(1)}) L_2(h_l^{(1)})}{h_l^{(1)} + h_n^{(1)}} ; \tag{4.4}$$

$$T_{nl}^{(12)} = \frac{iI_n}{\varepsilon A_l^{(2)} B_l^{(2)}} \frac{L_2(h_n^{(1)}) L_2(h_l^{(2)})}{h_l^{(2)} + h_n^{(1)}} ; \tag{4.5}$$

$$T_{nl}^{(13)} = I_n \frac{Z_2}{4k_1} A_l^{(3)} \varkappa_{1l}^{(3)} \varkappa_{2l}^{(3)} \sin(\varkappa_{1l}^{(3)} d) \sin[\varkappa_{2l}^{(3)}(d-D)] \frac{L_2(h_n^{(1)})}{L_2(h_l^{(3)})} \frac{1}{h_l^{(3)} - h_n^{(1)}} : \tag{4.6}$$

$$B_l^{(p)} = \begin{cases} (-1)^l h_l^{(p)} d_p & (l \neq 0) \\ 2k_p d_p & (l = 0) \end{cases} \quad (p = 1,\ 2) .$$

Making use of (3.7), we can obtain the following expressions for the squares of the moduli of the transformation coefficients:

$$|T_{nl}^{(1p)}|^2 = \left| \frac{4h_n^{(1)} h_l^{(p)}}{(h_n^{(1)} - h_l^{(p)})^2} \prod_{m=0}^{i^{(1)}}{}' \frac{1 + h_n^{(1)}/h_m^{(1)}}{1 - h_n^{(1)}/h_m^{(1)}} \prod_{m=0}^{i^{(2)}} \frac{1 + h_n^{(1)}/h_m^{(2)}}{1 \quad h_n^{(1)}/h_m^{(2)}} \prod_{m=0}^{i^{(3)}} \frac{1 \quad h_n^{(1)}/h_m^{(3)}}{1 \quad h_n^{(1)} + h_m^{(3)}} \times \right. \tag{4.7}$$

$$\times \prod_{m=0}^{i^{(1)}} \frac{1-h_l^{(p)}/h_m^{(1)}}{1+h_l^{(p)}/h_m^{(1)}} \prod_{m=0}^{i^{(2)}} \frac{1-h_l^{(p)}/h_m^{(2)}}{1+h_l^{(p)}/h_m^{(2)}} \prod_{m=0}^{i^{(3)}} \frac{1-h_l^{(p)}/h_m^{(3)}}{1+h_l^{(p)}/h_m^{(3)}} \Bigg| \qquad (p=1,\ 2,\ 3).$$

where

$$h_l^{(p)} = \begin{cases} -h_l^{(1,\,2)} & (p=1,\ 2) \\ h_l^{(3)} & (p=3) \end{cases}$$

and the factor with $m = l$ for $p = 1, 2, 3$ should be omitted.

The dependences of the square of the modulus of the transformation coefficient $|T_{00}|^2$ of the incident fundamental wave $n = 0$ in the first wave guide in the fundamental (slow) wave in the

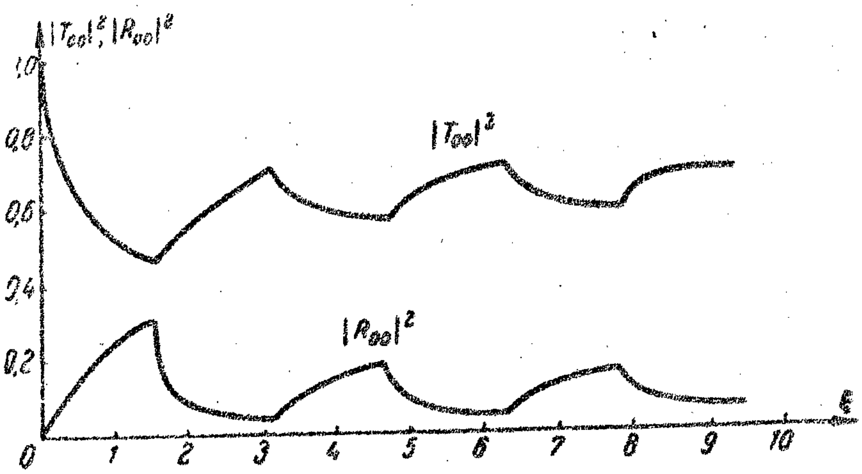

Fig. 4.

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second, and also the coefficient of reflection of the incident wave $|R_{00}|^2$ on the parameter $\xi = (D - d)/d$ are shown in Fig. 4 for values of the parameters given on page 7. Breaks in the curves take place at the critical values $\xi^{(p)}_{cr,m}$, which correspond to the appearance of alternate types of propagating waves in waveguides 2, 3.

For $\xi \to \infty$, the quantities $|T_{00}|^2 + |R_{00}|^2$ approach certain limiting values which can be approximately determined from Fig. 4: $|T_{00}|^2 \simeq 0.64$, $|R_{00}|^2 \simeq 0.1$. It is evident that these values characterize the transformation of energy of the incident ground fundamental wave into the energy of the surface wave on the dielectric plate and reflection of the incident wave from the open end of the waveguide in the unscreened system, respectively. The effectiveness of excitation of surface waves

$$\eta = \frac{|T_{00}|^2}{1 - |R_{00}|^2}$$

is seen to be approximately equal to 0.71 in the limit.

The same quantities will also apply to the system shown in Fig. 5a and representing a flat dielectric waveguide of thickness 2d, excited by a screened waveguide with a wave of the fundamental type. Thus, Eq. (4.7) can be used for the approximate calculation of the excitation of open dielectric waveguides.

It is interesting to note that only the wave numbers $h^{(p)}_m$ of the propagating waves enter into (4.7). A completely analogous

formula was obtained in [8] for the solution of the problem of the diffraction of electromagnetic waves on a shelf of surface impedance inside the waveguide. The method of solution considered here can evidently be applied to the calculation of the generation of slow H waves in the more general case also, in which the upper wall of the exciting waveguide is raised above the surface of the dielectric plate (Fig. 5b). In this case, the moduli of the transformation coefficients will again be described by Eq. (4.7). The problem is also solved for E waves by a similar method.

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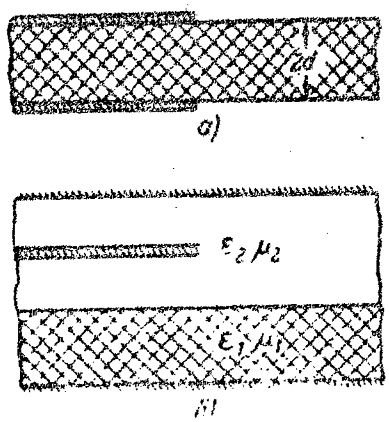

Fig. 5.

CONTROLLING AMPLITUDE AND PHASE OF ELECTROMAGNETIC WAVES IN A GUIDE BY MEANS OF A GERMANIUM PLATE*

[This is a translation of an article written by N. V. Aleksandrov, L. B. Gorskaya, E. M. Gershenzon and V. S. Etkin in Radiofizika (Radiophysics), Vol. II, No. 6, 1959, pages 911-914.]

(Submitted to editor June 8, 1959)

(Abstract) Possibility of control of the amplitude and phase of an electromagnetic wave in a waveguide by means of a germanium plate is demonstrated. The plate is placed perpendicular to the guide and in a magnetic field in the passage along it of the controlling current.

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*Paper given at the Third All-Union Conference MVO USSR on Radioelectronics, Kiev, 1959.

In the research of Gunn and Hogarth [1], an attenuator of centimeter waves was described which made use of the dependence of the absorption of electromagnetic waves by free carriers in

germanium on the concentration of the carriers. Since that time, this note has been frequently cited in reviews (see, for example, [3]), but new data on the use of suitable arrangements in the ultra-high frequency region have been produced. At the same time, this problem is of interest because the concentration of free carriers affects not only the conductivity, which determines the absorption of the electromagnetic waves, but also the dielectric constant of the semiconductor.

Actually, the conductivity and the dielectric constant of the semiconductor are determined by the expressions [4, 5]*

$$Z_{\text{вх}} = 1 - \operatorname{tg}\delta \, \frac{2\pi l \, [\varepsilon - (\lambda_0/\lambda_{\text{кр}})^2]}{\lambda \, [1 - (\lambda_0/\lambda_{\text{кр}})^2]} - j \, \frac{2\pi l \, |\varepsilon - 1|}{\lambda \, |1 - (\lambda_0/\lambda_{\text{кр}})^2|}$$

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*In the Gaussian system of units.

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where e is the charge on the electron, N is the number of current carriers per unit volume, τ is the time of free flight of the carriers, m^* is the effective mass of the carrier, ω is the electromagnetic field, ε_1 is the dielectric constant of the crystalline lattice, determined by the bound charges.

As in experimental investigations have shown [6, 7, 9], the contribution of free carriers to the dielectric constant of high resistance semiconductors can be very significant which makes it possible to use the measurements of the dielectric constant of semi-

conductors in the ultra-high-frequency region for the determination of m^* and τ.

In the present work, the possibility of phase and amplitude modulation of an electromagnetic wave incident on a germanium plate lying across the waveguide by means of a change in the concentration of free carriers has been studied.

Control of the concentration of carriers was achieved by use of the Hall effect in the germanium plate, where different rates of recombination were produced on opposite surfaces by scouring and mechanical polishing [1, 8]. The Hall emf compresses the line of the controlling current alternately to the differently treated surfaces, and by the same token changes the general conductivity of the plate as a result of the change of the mean lifetime of the current carriers [10]. The location of plate 1 in waveguide 2, and also the mutual orientation of the magnetic field H and the current j are shown in Fig. 1. High resistance ($\rho \sim$ 35-40 ohm-cm) n-type germanium was used in the research alloyed with antimony and having a concentration of free carriers of $N \sim 10^{14}$ particles/cm^{-3} and $\varepsilon_1 \sim 16$. The plates had a thickness $l = 0.25$-1mm.

The normalized input resistance of the plate, whose thickness is small in comparison with the electromagnetic wavelength in germanium loaded with a matched load, has the form [11]:

$$Z_{вх} = 1 - \tan\delta \; \frac{2\pi l[\varepsilon - (\lambda_0/\lambda_{кр})^2]}{\lambda[1-(\lambda_0/\lambda_{кр})^2]} - j\,\frac{2\pi l[\varepsilon - 1]}{\lambda[1-(\lambda_0/\lambda_{кр})^2]}$$

for $\tan \delta \ll 1$. Here λ_0 is the wavelength in free space, λ_{cr} is the critical wavelength, λ is the wavelength in the waveguide, $\tan \delta$ is the tangent of the loss angle in germanium.

One can derive the following expressions for the modulus q and the phase ϕ of the reflection coefficient:

$$q \simeq \frac{1}{\sqrt{1+|2/x'-\operatorname{tg}\delta|^2}};$$

$$\varphi = \operatorname{arc\,tg}\frac{-2}{x'-2\operatorname{tg}\delta} \simeq \operatorname{arc\,tg}\frac{-\lambda\,|1-(\lambda_0/\lambda_{\kappa p})^2|^2}{\pi l\,(\varepsilon-1)},$$

where

$$x' \doteq \frac{2\pi l\,|\varepsilon-1|}{\lambda\,|1-(\lambda_0/\lambda_{\kappa p})^2|}.$$

It is evident from this that the modulus and phase of the reflection coefficient depend upon ε. The corresponding dependence on ε and $\tan \delta$ also takes place for the transmitted wave.

The experimental investigation was carried out on apparatus whose block diagram is shown in Fig. 2. The phase and modulus of the reflection coefficient were measured on this apparatus, and also measurements were carried out on the phase of the transmitted wave as a function of the value of the current passing in the plate, and of the intensity of the magnetic field.

The results of the investigation reduce to the following.

The germanium plate in the magnetic field actually possesses a different conduttivity for currents of different directions (change in the direction of the current for fixed magnetic field changes the conductivity by a factor of 2). The modulus (Fig. 3a) and the phase

change (Fig. 3b) of the reflection coefficient are shown in Fig. 3, and also the phase change of the transmitted wave (Fig. 3c). The possibility of phase modulation whose intensity can be estimated from the difference in values of the ordinates for a given value of the current (for different directions of the current) follows from Figs. 3b and 3c.

If a section with a piston is placed at a distance of four wavelengths from the plate, instead of the equivalent antenna for the plate, then one can obtain a large phase modulation for small values of the reflection coefficient.

Modulation of the electromagnetic wave reflected by the germanium plate and passing through it was investigated by introduction on the plate of a variable stress. Modulation was accomplished by a sound generator or pulse generator. Dependence of the percentage modulation on the period of the modulating stress was not observed up to pulses up to length 0.1 microsecond. Oscillograms corresponding to a modulation frequency of 500 cycles are shown in Fig. 4 both for a reflected (Fig. 4a) and for the transmitted (Fig. 4b) wave for values of the intensity of the magnetic field of 1000 and 3000 oersteds and a matched load.

It should be noted that the increase in the difference between the rates of recombination on both surfaces as a result of a better treatment of the plate should materially increase the effectiveness of the modulator described.

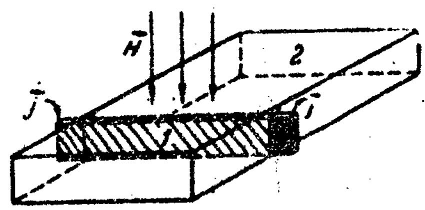

Fig. 1.

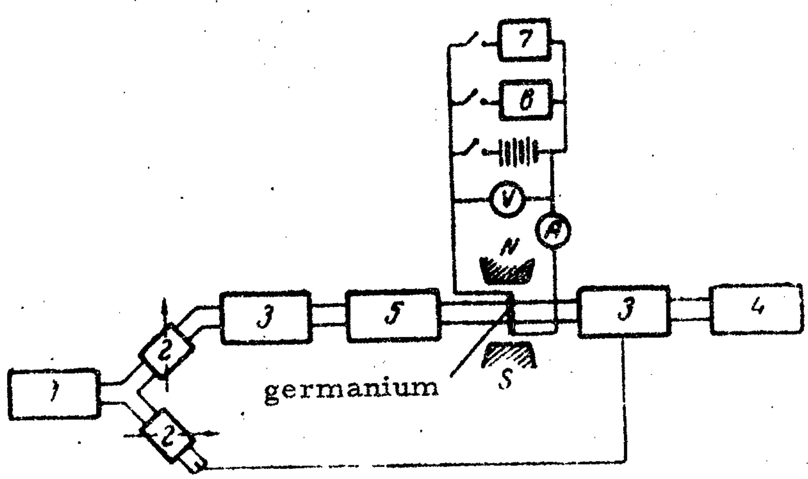

Fig. 2. Block diagram of the apparatus

1--generator, 2--attenuators, 3--measuring line, 4--equivalent antenna, 5--measuring line, 6--sound generator, 7--pulse generator.

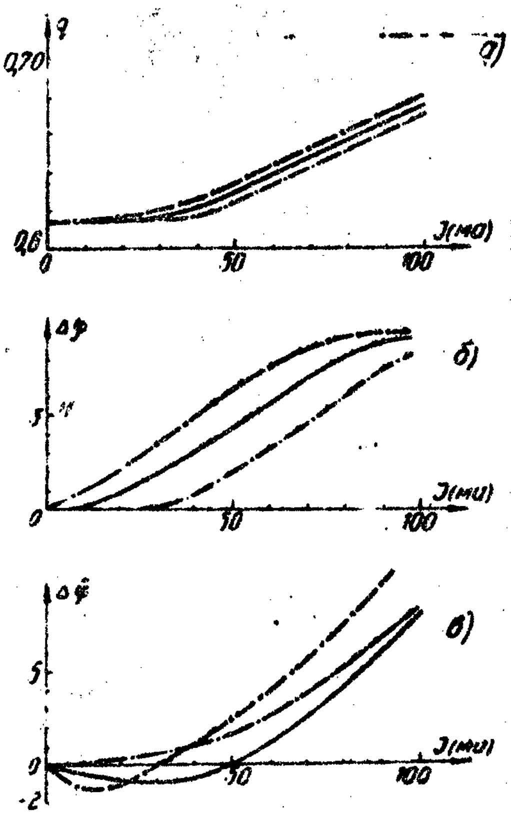

Fig. 3. Dependence of the modulus of the reflection coefficient, change in phase of the reflected ($\Delta\ \phi$) and transmitted ($\Delta\ \bar{\Phi}$) wave on the current I through the germanium plate:

--in the absence of a magnetic field, --x--, --·-- in the presence of a magnetic field and for different directions of the current

(1 = 0.95 mm).

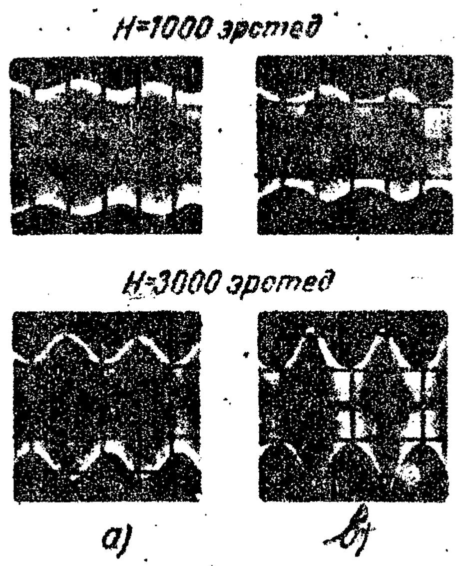

Fig. 4. The Russian word in both halves of the figure is oersted

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INVESTIGATION OF THE FLUCTUATIONS OF OSCILLATION OF A KLYSTRON GENERATOR*

[This is a translation of an article written by V. N. Nikonov in Radiofizika (Radiophysics), Vol. II, No. 6, 1959, pages 914-926.]

(Submitted to editor May 17, 1959)

(Abstract) The effect of the interference of the shot effect in current induced in a resonator in the superposition of counterflows of electrons on the fluctuation of frequency and amplitude of vibrations is considered. It is shown that, as a consequence of the interference of the shot effect, the line width, and spectral density of the fluctuations of the amplitude of the vibrations for voltage applied to the reflector, increased in comparison with the optimum, and exceeds by a factor of 1.5-3 the line width and spectral density of fluctuations of amplitude for decreased voltage, under definite conditions. The effect of flicker noise on the line width of the vibrations as also in the vacuum tube generator, cannot be obtained from a

solution of the first approximation of the nonlinear problem. An experimental investigation is carried out of the "natural" fluctuations of the oscillations of a klystron at a wavelength of 3.2 cm as a function of the oscillation states, which shows good agreement with theory. For the usual oscillation states, the measured spectral line width corresponds to 0.1-0.5 cycles; the band of the spectrum of amplitude fluctuations changes from 20 to 50 megacycles, while the spectral density changes from 10^{-16} to 10^{-15} cycles^{-1}.

*The results of the research were reported at the Conference on Statistical Radiophysics (Gorky, 1958).

A number of researches have been devoted to the theoretical and experimental investigation of the fluctuations of the oscillations of a klystron generator [1-3]. However, several important facets of this problem remain unclear both theoretically and experimentally. The present research fills in this problem in definite measure. In the theoretical part of the paper, the effect is considered of shot and flicker effects on the flow of electrons on the amplitude and frequencies of vibration. Account of the effect of amplitude fluctuations on the frequency of the generator is also carried out in comparison with [1], and also account of the interference of the Fourier components of the shot noise arising in the resonator in the superposition of random components of the current

of direct and inverse electron flows. Consideration is carried out by the spectral method [4]; as initial equations, the equations of the klystron generator obtained in reference [1] are used.

The first experimental investigations of "natural" amplitude and frequency fluctuations of the klystron oscillations were completed in reference [2]. However, only single measurements were made in [2], which cannot give a complete picture of the behavior of the fluctuations. In the present paper, a method is set forth which makes it possible to carry out measurements of amplitude fluctuations; the results of a detailed investigation of "natural" amplitude and frequency fluctuations of the klystron oscillations as a function of the regimes of its operation are set forth and compared with theory.

1. THE THEORY OF OSCILLATION FLUCTUATIONS OF THE KLYSTRON GENERATOR

According to [1], in the case of idealization of the cavity of a klystron by an L, C and R circuit, the dynamic equations of the system can be written in the forms:

$$L\frac{di}{dt} = v - Ri; \tag{1}$$

$$C\frac{dv}{dt} = -i + I_0\left[\frac{1}{1 + (\tau_0\, 2U_a)\, dv_1\, dt_1} - 1\right], \tag{1a}$$

where L, C and R are the equivalent parameters of the oscillation system of the klystron, v is the variable voltage on the capacitor, i is the oscillation current in the circuit. The second term in Eq. (1a) represents the current carried by the electron flux penetrating

the grid (I_0 is the cathode current of the tube, τ_0 is the mean time of flight of the electron from the cavity to the reflector and return, U_a is the constant voltage on the cavity of the klystron). In the derivation of Eqs. (1)-(1a), the time of flight of the electron through the gap between the grids of the cavity is neglected.

The beam of electrons penetrates the grids of the cavity twice. The time t of penetration of the grid of the cavity by an electron after reflection is connected with the time of passage by this same electron through a grid toward the reflector d_1 by the relation

$$t = t_1 + \tau_0 + (\tau_0/2U_a)v_1,$$

where v_1 is the voltage between the grids at the time t_1.

In the consideration of the fluctuations of the oscillations, we shall consider as the source of these fluctuations not nnly the shot effect but also the flicker effect in the electron beam. The thermal effect can be neglected in the case under consideration [1].

Designating by $f_1(t)$ and $f_2(t)$ the random components of the induced current produced by the shot and flicker effects of the beam of electrons penetrating the grids of the cavity, and by $f_1'(t)$ and $f_2'(t)$ the components of the current induced by the reflected beam, we write down Eqs. (1)-(1a) with account of random currents:

$$L\frac{di}{dt} = v - Ri; \tag{2}$$

$$C\frac{dv}{dt} = -i + I_0\left[\frac{1}{1 + (\tau_0/2U_a)\,dv_1/dt_1} - 1\right] - \tag{2a}$$

$$-\left[f_1(t) + f_2(t) + f_1'(t) + f_2'(t)\right].$$

Equations (2)-(2a) differ from the equations used in reference [1] only by the consideration of the flicker effect in the electron beams. However, the spectral approach [4] to the solution of these equations gives results which differ somewhat from those obtained in [1]. Therefore, we shall carry out a detailed solution of Eqs. (2)-(2a).

The random components of the current induced by the reflected beam (which represents an alternation of compression and rarefaction of the electrons), are periodic non-stationary functions and are expressed in terms of the stationary components of the current at the time of passage through the grid to the reflector, t_1, by the following relations:

$$f_1'(t) = \frac{f_1(t_1)}{1 + (\tau_0/2U_a)\, dv_1\, dt_1}; \qquad f_2'(t) = \frac{f_2(t_1)}{1 + (\tau_0/2U_a)\, dv_1\, dt_1}$$

Transforming in (2)-(2a) to polar coordinates r and θ

$$\sqrt{L/C}\, i = r\cos\vartheta, \quad v = r\sin\vartheta$$

and using the method of averaging over θ [4], we obtain:

$$\frac{dr}{dt} = \Phi(r) + \frac{1}{2\pi C}\int_0^{2\pi}\Big[f_1(t) + f_2(t) - \frac{f_1(t_1) + f_2(t_1)}{1 - (\omega_0\tau_0/2U_a)\, r\cos\vartheta_1}\Big]\sin\vartheta\, d\vartheta; \tag{3}$$

$$\frac{d\vartheta}{dt} = -\omega_0 + \Psi(r) - \frac{1}{2\pi C}\int_0^{2\pi}\Big[f_1(t) + f_2(t) - \frac{f_1(t_1) + f_2(t_1)}{1 - (\omega_0\tau_0/2U_a)\, r\cos\vartheta_1}\Big]\cos\vartheta\, d\vartheta, \tag{3a}$$

where, according to [1],

$$\Phi(r) = -\alpha r - (I_0/C)\sin(\omega_0\tau_0) J_1(Xr);$$

$$\Psi(r) = (I_0/Cr)\cos(\omega_0\tau_0) J_1(Xr). \tag{3b}$$

In the given expressions $\omega_0 = 1/\sqrt{LC}$, $\alpha = R/2L$, $J_1(Xr)$ is the Bessel function of first order with argument $Xr = (\omega_0\tau_0/2U_a)r$.

The stationary amplitude of the auto-oscillations is determined by the condition $\Phi(r) = 0$ for $p = -(d\Phi/dr) > 0$. The quantity $\Psi(r_0)$ characterizes the non-isochronous character of the generator. In the maximum of the region of generation, where $\Psi(r_0) = 0$, the generator can be regarded as isochronous. The presence of fluctuating forces in Eqs. (3)-(3a) leads to random modulation of amplitude and phase of the oscillations of the form $r = r_0[1 + \alpha(t)]$ and $\theta = \theta_0 + \Phi(t)$, where $\alpha(t)$ is the random percentage modulation of the amplitude and $\Phi(t)$ is the phase of oscillation. Setting $\alpha(t) \ll 1$ and $d\theta_0/dt = -\omega_0 + \Psi(r)$, and carrying out linearization of Eqs. (3)-(3a) in $\alpha(t)$, we obtain equations for the fluctuations of amplitude and phase of the oscillations:

$$\frac{d\alpha}{dt} = -p\alpha(t) + \frac{1}{2\pi r_0 C}\int_0^{2\pi} [f_1(t) + f_2(t)]\sin\vartheta\, d\vartheta - \\ -\frac{1}{2\pi r_0 C}\int_0^{2\pi} [f_1(t_1) + f_2(t_1)]\sin[\vartheta_1 - \omega_0\tau_0 - Xr_0\sin\vartheta_1]\, d\vartheta_1; \tag{4}$$

$$\frac{d\varphi}{dt} = \nu(t) = qr_0 a(t) - \frac{1}{2\pi r_0 C}\int_0^{2\pi} [f_1(t) + f_2(t)] \cos\vartheta \, d\vartheta + \tag{4a}$$

$$\frac{1}{2\pi r_0 C}\int_0^{2\pi} [f_1(t_1) + f_2(t_1)] \cos[\vartheta_1 - \omega_0\tau_0 - Xr_0 \sin\vartheta_1] \, d\vartheta_1,$$

where $p = p(r_0) = (d\Phi/dr)_{r=r_0}$ and $q = q(r_0) = (d\Psi/dr)_{r=r_0}$. It can be shown from (3b) that p and q are connected by the relation $qr_0 = -p \cot(\omega_0 \tau_0)$. In the second, fluctuating components of (4) and (4a), the substitution of the variable θ for θ_1 has been carried out.

We shall compute the integrals lying on the right hand sides of Eqs. (4)-(4a). For this purpose, we expand the function in a Fourier series

$$\sin[\vartheta_1 - \omega_0\tau_0 - Xr_0 \sin\vartheta_1] = a_0 2 + \sum_{n=1} [a_n \sin(n\vartheta_1) + b_n \cos(n\vartheta_1)];$$

$$\cos[\vartheta_1 - \omega_0\tau_0 - Xr_0 \sin\vartheta_1] = c_0 2 + \sum_{n=1}^{\infty} [c_n \sin(n\vartheta_1) + d_n \cos(n\vartheta_1)].$$

Substituting these expansions in Eqs. (4)-(4a), we note that the integrals on the right hand side will be different from zero if the integrand functions change slowly in comparison with sin θ and cos θ. The latter is the case if there are frequencies in the spectra of the random quantities which are close or equal to the frequency of generation or its higher harmonics. Assuming also that the spectrum of the flicker effect extends to zero frequency and that this effect can be neglected in the region of radio frequencies, we obtain:

$$\frac{d\alpha(t)}{dt} = -p\alpha(t) + \frac{f_1(t)}{r_0 C}\sin\vartheta - \frac{f_1(t_1)}{r_0 C}\Big\{a_0 2 +$$

$$+ \sum_{n=1} [a_n \sin(n\vartheta_1) + b_n \cos(n\vartheta_1)]\Big\} - \frac{a_0 f_2(t_1)}{2r_0 C}; \qquad (5)$$

$$\frac{d\varphi(t)}{dt} = \nu(t) = \Delta\alpha(t) - \frac{f_1(t)}{r_0 C}\cos\vartheta + \frac{f_1(t_1)}{r_0 C}\Big\{c_0 2 +$$

$$+ \sum_{n=1}^{\infty} [c_n \sin(n\vartheta_1) + d_n \cos(n\vartheta_1)]\Big\} - \frac{c_0 f_2(t_1)}{2r_0 C}, \qquad (5a)$$

where the coefficient $\Delta = qr_0$ shows how much the oscillation frequency changes upon doubling the amplitude. It is not difficult to see that a_0 and c_0 are proportional to the amplitude of the fundamental of the induced current.

Solving the resultant equations by thespectral method and replacing the coefficients a_0, c_0, a_n, b_n, c_n, d_n by their explicit expressions, we finally obtain the following expressions for the spectral densities of the fluctuations of amplitude and frequency of the oscillations:

$$w_\alpha(\Omega) = \frac{w_{f_1}}{r_0^2 C^2}\frac{1}{p^2+\Omega^2}\Big\{1 - \frac{1}{2}\cos(2\omega_0\tau_0) J_2(2Xr_0) -$$

$$- \cos(\omega_0\tau_0)[J_0(Xr_0) - J_2(Xr_0)]\Big\} + \frac{w_{f_2}(\Omega)}{r_0^2 C^2 (p^2+\Omega^2)}\sin^2(\omega_0\tau_0) J_1^2(Xr_0); \qquad (6)$$

$$\widetilde{w}_\nu(\Omega) = \Big\{\frac{w_{f_1}}{r_0^2 C^2}\Big[1 + \frac{1}{2}\cos(2\omega_0\tau_0) J_2(2Xr_0) - \cos(\omega_0\tau_0) J_0(Xr_0) -$$

$$
- \cos(\omega_0\tau_0) J_2(Xr_0)\Big] + \frac{w_{f_2}(\Omega)}{r_0^2 C^2}\cos^2(\omega_0\tau_0) J_1^2(Xr_0)\Big\} + \{\Delta^2 w_\alpha(\Omega)| + \tag{6a}
$$

$$
+ \left\{ \frac{w_{f_1}}{r_0^2 C^2} \frac{\Delta p}{p^2+\Omega^2}\Big[\sin(2\omega_0\tau_0) J_2(2Xr_0) - 2\sin(\omega_0\tau_0) J_2(Xr_0)\Big| + \right.
$$

$$
\left. + \frac{w_{f_2}(\Omega)}{r_0^2 C^2} \frac{\Delta p}{p^2+\Omega^2}\sin(2\omega_0\tau_0) J_1^2(Xr_0) \right\}.
$$

H~~Wh~~ere w_{f_1} is the spectral density of the shot effect, $w_{f_2}(\Omega)$ is the spectral density of the flicker effect, $J_0(Xr_0)$, $J_1(Xr_0)$ and $J_2(2Xr_0)$ are the Bessel functions of the first kind of zeroth, first and second order.

In the expression (6a) for the spectral density of the fluctuations of the frequency of oscillation, the terms in the first curly brackets describe the direct effect of the shot and flicker effects on the frequency of oscillation, while those in the second curly brackets the effect of the non-isochronous character of the generator, and the third curly bracket owes its appearance to the correlation of the direct fluctuations of the frequency with fluctuations from the non-isochronous behavior of the generator.

If we neglect the non-isochronous behavior of the generator (which can be done when the klystron operates in regimes close to the maximum of the region of generation) and limit ourselves to a consideration of the action of the shot processes alone, then we obtain the following expressions for the spectral densities of the fluctuations of amplitude and frequency:

$$w_\alpha(\Omega) = \frac{1}{p^2+\Omega^2}\left\{\frac{eI_0}{r_0^2C^2} + \frac{eI_0}{r_0^2C^2}\left[1 - \cos(2\omega_0\tau_0)\, J_2(2Xr_0)\right] - \right. \tag{7}$$

$$\left. - \frac{2eI_0}{r_0^2C^2}\cos(\omega_0\tau_0)\left[J_0(Xr_0) - J_2(Xr_0)\right]\right\};$$

$$w_\alpha = \frac{eI_0}{r_0^2C^2} + \frac{eI_0}{r_0^2C^2}\left[1 + \cos(2\omega_0\tau_0)\, J_2(2Xr_0)\right] - \tag{7a}$$

$$- \frac{2eI_0}{r_0^2C^2}\cos(\omega_0\tau_0)\left[J_0(Xr_0) + J_2(Xr_0)\right].$$

Here it has been taken into account that in the latter case $w_{f_1} = 2eI_0$(e is the charge on the electron). These expressions differ by the presence of a third component from the similar relations obtained in the research of [1]. The first components in both expressions represent the contribution to the fluctuation of the amplitude and frequency of the shot effect in the electron beam; these first penetrate the cavity on the way to the reflector. The second components characterize the contribution of the shot effect of the reflected electron beam. The third component is connected with the

interferences of each of the Fourier components of the shot effect; this interference arises in the cavity upon superposition of the random components of the current of direct and reflected beams. The interference term vanishes at the maximum of the region of generation under the condition $\omega_0\tau_0 = (n + 3/4)2\pi$. For deviation in the direction of smaller or larger transit angle $\omega_0\tau_0$, it becomes respectively positive or negative, bringing about at the same time a non-symmetrical of w_a and w_ν in the range of electron tuning. For detuning relative to the maximum in the currents, where the power of the generator amounts to one-half of the power in the maximum region, w_a and w_ν for smaller transit angles (by a factor of 1.5-3) will exceed w_a and w_ν on the other side of the region of generation. The difference in the change of w_a for change in the transit angle comes about as an interference term and also from the asymmetries of the region of generation. The asymmetry of the region of generation also affects the difference in the change of the total spectrum of the frequency fluctuations $\tilde{w}_\nu$ in terms of its component $\Delta^2 w_\alpha$.

The expression (7)-(7a) can be transformed to another form by replacing C and r_0^2 by $Q/\omega_0 Z_p$ and $P2Z_p$ (Q is the quality factor of the cavity, Z_p is the resonance resistance of the circuit, P is the power of the oscillation). We then obtain the result that

$$w_\alpha(\Omega) = \frac{\omega_0^2}{Q^2}\frac{eI_0Z_p}{P}\frac{1}{p^2+\Omega^2}\left\{1 - \frac{1}{2}\cos(2\omega_0\tau_0)J_2(2Xr_0) - \cos(\omega_0\tau_0)\left[J_0(Xr_0) - J_2(Xr_0)\right]\right\}; \tag{8}$$

$$w_\nu = \frac{\omega_0^2}{Q^2}\frac{eI_0 Z_p}{P}\Big\{1 + \frac{1}{2}\cos(2\omega_0\tau_0) J_2(2Xr_0) -$$

$$- \cos(\omega_0\tau_0)\left[J_0(Xr_0) + J_2(Xr_0)\right]\Big\}. \tag{8a}$$

Taking it into account that $\omega_0/Q = 2\pi\Delta f$, it is not difficult to see that the spectral densities of the fluctuations of frequency and amplitude w_ν, w_a, are proportional to the ratio of the power of the shot noise in the cavity to the power of the oscillations. As is known [5], the quantity $w_\nu(0)/4\pi$ in the case of the consideration of the shot effect alone determines the "natural width" of the lines of oscillation, measured in cycles.

We shall consider in more detail the flicker effect of the beam of electrons on the fluctuations of amplitude and frequency. Neglecting in (6)-(6a) components associated with the shot noise, and assuming that

$$\alpha = R/2L = \omega_0/2Q,\ \sin(\omega_0\tau_0) J_1(Xr_0) = C\omega_0 r_0/2QI_0,\ w_{I_2}(\Omega) = I_0^2\eta(\Omega),$$

we obtain

$$w_a(\Omega)_\Phi = \frac{\omega_0^2\eta(\Omega)}{4Q^2(p^2+\Omega^2)};$$

$$w_\nu(\Omega)_\Phi = \frac{\omega_0^2\eta(\Omega)}{4Q^2}\left[\operatorname{ctg}^2(\omega_0\tau_0) + \frac{\Delta^2}{p^2+\Omega^2} + \frac{2\Delta p}{p^2+\Omega^2}\operatorname{ctg}(\omega_0\tau_0)\right]. \tag{9}$$

Replacing Δ in this expression by its equivalent in terms of p, we get for the fluctuations of frequency:

$$w_\nu(\Omega)_\Phi = \frac{\omega_0^2\eta(\Omega)}{4Q^2}\frac{\Omega^2}{p^2+\Omega^2}\operatorname{ctg}^2(\omega_0\tau_0). \tag{10}$$

Assuming that the spectrum $\eta(\Omega)$ has the form: $\eta(\Omega) = n/\Omega^\lambda$, (where m and λ are quantities independent of Ω), it is not difficult to see that for $\lambda \leq 2$, the spectral density of $w_\nu(\Omega)_\Phi$ as $\Omega \to 0$ falls or

remains constant, while the spectral density $w_{\alpha}(\Omega)_{\Phi}$ under these conditions increases without limit. If, in accord with [6], we take it into account that for tubes with oxide cathodes, $\bar{m} \sim 10^{-12}$ cycles$^{\lambda-1}$ and $1 \leq \lambda \leq 2$, then $w_{\nu}(\Omega)_{\Phi} \sim 10^{-9}$ rad^2-sec^{-1} (for $\lambda = 2$; $\Omega = 2 \times 10^3$; $f = 9 \times 10^9$) will be much smaller than the spectral density of the frequency fluctuations which arise because of the shot effect. We note that $w_{\alpha}(\Omega)_{\Phi}$ for $m \sim 10^{-12}$, $\lambda = 2$ and $p = 10^7$ cycles will be equal in order of magnitude to w_{α} from the shot effect at a frequency of 40 cycles.

Thus, in the given approximation, account of the flicker effect in the klystron (just as in a vacuum tube generator [7]) does not explain the increase in the spectral density of the frequency fluctuations in the direction of low frequencies.

2. EXPERIMENTAL STUDY OF THE FLUCTUATIONS OF OSCILLATION OF A KLYSTRON GENERATOR

The experimental arrangement for the investigation of the fluctuations of the oscillation of a klystron was similar to that used in reference [2]. A block diagram of the apparatus is shown in Fig. 1. The oscillations under investigation fell on a crystal detector or simultaneously by means of a short channel and delay line, or only by a short channel. The order of measurement of noise at the output of the detector, determined by the fluctuations of frequency, was the same as in reference [2]. At first the noise was measured in the interference of two oscillations traveling along different channels to

the crystal. For a definite phase difference ψ_0 between their amplitudes, the vibrations were so selected that they formed (jointly with difference in amplitude) a right triangle [5, 8]. In this case, noises were measured which were determined by phase and amplitude fluctuations, plus the self noise of the detector. In the feeding of oscillations by one channel and by the equality of its amplitude to the amplitude of the difference oscillation of the first measurement, noises were measured which were determined by amplitude fluctuations and by the self noises of the detector. The difference between the results of the two measurements gave the noise at the output of the detector, determined by fluctuations of the frequency of vibration.

The self-noise of the detector, which is necessary for determination of amplitude fluctuations, was measured by means of a special circuit of suppression of amplitude fluctuations on a twin T with two oppositely connected crystals (similar to the circuit usually employed for noise suppression of a heterodyne in uhf receivers). Balancing of such a circuit was carried out by amplitude modulation of the klystron oscillations with a signal having a frequency falling in the band of the fluctuation meter; this signal was in turn modulated by an acoustic frequency. The regime of the detectors was brought into alignment by ear by the total suppression of amplitude modulation. The noises measured in such a mixing are the self noise of two crystals switched in in the particular manner

(Fig. 2). Then the output of the crystals was connected in cophasal fashion. In this case the noise at the output of the detectors was generated from the known noise of the crystals and the noise determined by the amplitude fluctuations. By subtraction of the result of the first measurement from the result of the second measurement, the spectral density of noise was obtained, connected with the amplitude fluctuations. We have assumed that in the switching over of the output of the crystals, their resulting self noise undergoes little change.

Measurement of the noise of the output of the detector was carried out by comparing it with the noise voltage of the diode to D to S connected to the input of the fluctuation meter. All the rest of the apparatus in this case served as a zero indicator of the equality of the mean square of both noise voltages in the band of the fluctuation meter. Analysis of the method of measurement is given in detail in references [5, 8]. In the treatment of the results of the measurements, the formulas for the spectral density of noise power at the output of the detectors W_α (in the measurement of amplitude fluctuations) and W_ν (in the measurement of phase fluctuations) which are given in [5] were employed:

$$W_\alpha = \beta_1^2 V^2 w_\alpha (F); \tag{11}$$

$$W_\nu = \beta^2 V_0^2 \tau^2 \cos^2\Psi_0 \frac{\sin^2(\Omega\tau\ 2)}{(\Omega\tau\ 2)^2} \left| w_\nu + \Delta^2 w_\alpha + \Omega \operatorname{ctg}(\Omega\tau\ 2) \operatorname{tg}\Psi_0 \Delta w_\alpha \right|, \tag{12}$$

where β and β_1 are the transfer coefficients of the detector for measurement of frequency and amplitude noises, respectively, V_0 is the amplitude of the oscillation at the detector which enters along the short waveguide channel, τ is the delay time, $\Omega = 2\pi F$ is the frequency determined by the spectrum-analyzer.

We designate the quantity appearing in square brackets in (12) by the letter Θ_ν. The third term in Θ_ν characterizes the additional component in the noise spectrum at the output of the detector which appears as a result of the correlation part of the frequency fluctuations with the amplitude fluctuations. Depending on the signs of tan ψ_0 and Δ, this correlation term can enter into the expression for W_ν with a plus or minus sign. As a consequence of the dependence on the actual frequency and parameters of the delay line, the presence of the additional components at the output of the detector will appear in a strong and sometimes unusual dependence of the spectral density on the frequency. Curve IV in Fig. 3 can serve as an example of such an unusual frequency dependence (increase with increase in frequency); here the dependence of the spectral density $\Theta_\nu = \tilde{w}_\nu + W(\Omega, l)$ on the frequency F is shown.

It is easy to see that in the measurement of the noise spectrum in an unlimited range of frequencies it is not possible to obtain attenuation of the correlation components over the entire range investigated by a choice of the delay time τ. An exception is the case

of its practical absence at the output of the detector. For the determination of the frequency intervals in which one can neglect this additional spectrum for a given delay time, one must satisfy the inequality following from the results of [5]:

$$\operatorname{tg} \Psi_0 \operatorname{ctg} (\Omega\tau/2) \ll \Delta/\Omega. \tag{13}$$

This inequality is obtained from the condition

$$w_\nu + \Delta^2 w_\alpha \gg \Omega \operatorname{ctg} (\Omega\tau/2) \operatorname{tg} \Psi_0 \Delta w_\alpha ,$$

where $w_\nu \ll \Delta^2 w_a$. The intervals shown will be located around the frequencies $\Omega = \pi(2n + 1)/\tau$, in which the correlation term vanishes.

Exclusion of the correlation component can be brought about with the help of the repeated interference measurement with a choice of the angle Ψ between the voltage vectors equal to $2\pi - \Psi_0$. The latter is achieved by a change in the length of the delay line. As a result there is a change of sign for the correlation components of the spectral density Θ_ν. After superposition of the results of the first and repeated measurements and division by 2, we obtain: $\tilde{w}_\nu = w_\nu + \Delta^2 w_a$. The difference between these results of measurement makes it possible to estimate the coefficient Δ.

3. RESULTS OF MEASUREMENT

The klystrons were studied at a wavelength of 3.2 cm in an operating regime corresponding to $6(\omega_0 \tau_0 = 13.5\pi)$ and 7 $(\omega_0 \tau_0 = 15.5\pi)$ numbers of regions of generation. The length of the delay line was 14.4 m. The range of frequencies investigated

was from 0.1 to 30 Mc in the case of amplitude fluctuations and from 0.1 to 10 Mc in the case of frequency fluctuations.

The measurements were carried out in the maximum region of generation (curve 1 in Figs. 3-5; all the drawings refer to a klystron of a single type), and also for detuning relative to the maximum in the direction of increase (curve II) and the direction of decrease (curve III) in the angle of flight (for a power of generation equal to 0.56 of the maximum power).

The quantities r_0 and p, which are required for the calculation of the spectral densities of fluctuations w_α and w_ν were determined by several simple measurements and calculation on the basis of the results of [1]. The capacitance C which appears in Eqs. (7)-(7a) was estimated from measurements of Z_p with the help of the relation $C = Q/\omega_0 Z_p$. In this case, the quality factor of the cavity Q was assumed known from cold measurements. For $Q = 2 \times 10^3$, the capacitance of the gaps for different klystrons lay in the limits from 1.8 to 2.5 mm.

<u>Amplitude fluctuations</u>. After treatment by Eq. (11) of the results of measurement of the noise generated at the output of the detector by amplitude fluctuations, one can find the spectral density of the percentage random modulation of the oscillations w_α. The experimental curves for the dependence of the spectral density w_α on frequency F (for the region of generation for $\omega_0\tau_0 = 13.5\pi$, they are shown in Fig. 4) are satisfactorily approximated by

functions of the form $w_a = B[1 + (F/p)^2]^{-1}$, where B determines the intensity and p the band of amplitude fluctuations.

The results of the measurements of the spectral densities of the amplitude fluctuations are given in Table 1; the values given here correspond to the values computed by Eq. (7) for C = 2mmf, $U_a = 300v$, $I_0 = 25ma$. The results given in Table 1 confirm the very well known fact that in the detuning on both sides of the maximum of the region of generation (to the half-power point) the value of B increases while the band p decreases more rapidly in the case of detuning with decreased angle of flight than with increase in it. This phenomenon finds its explanation in the asymmetry of the region of generation. In the case of a mirror shift of the asymmetry, the situation is changed: B and p will change more rapidly with increase in the angle of flight.

We note that the noise power of the sidebands of amplitude modulation can be estimated by the formula $P_n = \overline{a}^2 P_A$, where P_A is the power in the generation line, while $\overline{a}^2$ is the mean square of the percentage modulation, equal to $\int_0^\infty w_a(F)dF = \pi Bp/2$. Thus, for example, for the curve I of Fig. 4, $\overline{a}^2 = 1.7 \times 10^{-8}$ (i.e., $\sqrt{\overline{\alpha^2}} = 0.013$ percent).

<u>Frequency fluctuations</u>. Using (12) and the results of measurements of the noise spectra at the output of the detector in an interference experiment, we obtain curves for the spectral density Θ_ν given in Fig. 3. These curves contain the spectrum of the

frequency fluctuations $\tilde{w}_\nu$ and the additional correlation spectrum mentioned above. Taking into account the fact that the correlation spectrum vanishes for $F = 1/2\tau$ (in our case for $F = 7.5$Mc) and that the spectrum of the amplitude fluctuations differs little in the range from 0.1 to 10Mc, we can separate the correlation spectrum from Θ_ν and determine the values of the quantities $\tilde{w}_\nu$ and Δw_a. The spectral density of the frequency fluctuations $\tilde{w}_\nu(F)$ is given in Fig. 5. Knowledge of w_a from measurements of the amplitude fluctuations make it possible to determine the coefficient Δ, and then to estimate the value of the components of the spectral density of the frequency fluctuations $\Delta^2 w_a$ and the remainder w_ν. Detailed procedure makes it possible to carry out a separation of the components of spectral density Θ_ν for the case of the region of generation; however, in operation in the maximum the correlation noise is small and reliable determination of the coefficient proves to be impossible.

Table 2 gives the experimental values of the quantities $\tilde{w}_\nu$, Δ, w_ν; corresponding values of w_ν computed from Eq. (7a) are given for comparison.

The results of measurements of the spectral density of frequency fluctuations of the klystron $\tilde{w}_\nu$, given in Table 2, make it possible to determine the quantity "natural width" of lines of oscillation of the klystron ΔF and to explain the behavior of this quantity upon change in the operating regime.

For the maximum of the region of generation (curve 1 in Fig. 5), the "natural width" of the line $\Delta F = 0.08$cps. The theoretical value of the line width for the same parameters of the generator as in the case of calculation of amplitude fluctuations is equal to 0.087 cps; the relative line width is $\Delta F/F \sim 10^{-11}$. In the transition to higher numbers of the regions, when the angle of flight is increased, the line width of the oscillations of the klystron generator increase approximately in the same ratio as the quantity which is the reciprocal of the power generated. For change in the angle of flight the process of electron tuning from the center of the region to the half-power point the line width increases by a factor of 1.5-2 upon increase and 2.4 upon decrease of the angle of flight.

Thus the spectral line width for electron tuning changes unsymmetrically. Unsymmetrical change of the line width of the generation is a consequence of the lack of symmetry of the change of both components of the spectral density $\tilde{w}_\nu = w_\nu + \Delta^2 w_a$ (see Tables 1 and 2). The nonsymmetrical change of w_ν is possibly the result of the interference of the random components of the shot noise in the cavity in the superposition of the direct and inverse electron beams.

As follows from what has been said above, experiments on the measurement of frequency and amplitude fluctuations of klystron oscillations give satisfactory agreement with the theoretical investigation of the fluctuations of the oscillator vibrations.

In conclusion I express my deep gratitude to V. S. Troitskii

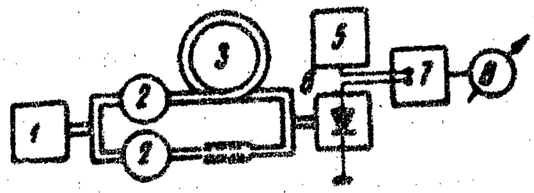

Fig. 1. Block diagram for measurement of the fluctuations of oscillations of the generator: 1-generator, 2-attenuator, 3-delay line, 4-line of variable length, 5-noise diode, 6-detector, 7-null fluctuometer, 8-output operators.

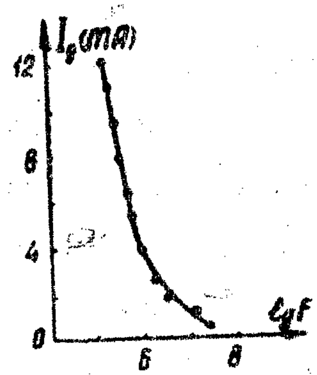

Fig. 2. Frequency spectrum of the noise of a crystal detector: noise current of the diode J_d is plotted along the ordinate)

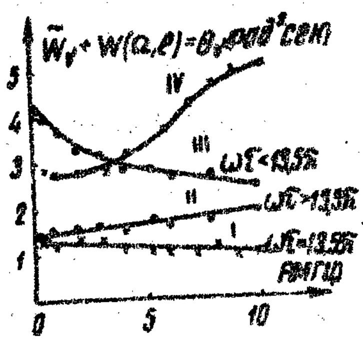

Fig. 3. Dependence of the spectral density of frequency fluctuations of vibrations with unseparated correlation components $\theta_{\perp} = \tilde{w}_{\nu} + W(\Omega, \ell)$ on the frequency F for klystron no. 1: I-- $\omega\tau = 13.5\pi$; II-- $\omega\tau > 13.5\pi$; III-- $\omega\tau < 13.5\pi$.

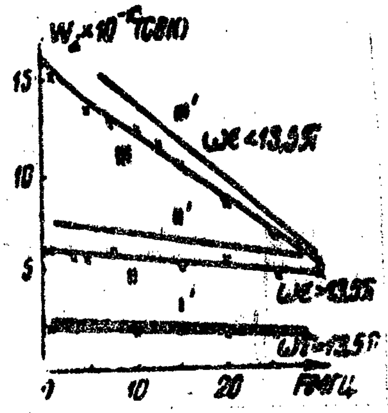

Fig. 4. Dependence of the spectral density of the amplitude fluctuations of the oscillations w_a on the frequency F for klystron No. 1 (I, II, III are experimental curves, Ia, IIa, IIIa are theoretical curves): I-- $\omega\tau = 13.5\pi$; II-- $\omega\tau > 13.5\pi$; III-- $\omega\tau < 13.5\pi$.

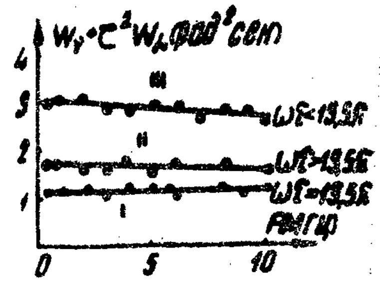

Fig. 5. Dependence of the spectral density of frequency fluctuations of the oscillations w_ν on the frequency F for klystron No. 1: I-- $\omega\tau = 13.5\pi$; II-- $\omega\tau > 13.5\pi$; III-- $\omega\tau < 13.5\pi$.

Table 1

		$\omega_0\tau_0=13.5\pi$	$\omega_0\tau_0=13.5\pi$	$\omega_0\tau_0=13.5\pi$	$\omega_0\tau_0=15.5\pi$	$\omega_0\tau_0=15.5\pi$	$\omega_0\tau_0=15.5\pi$
Klystron № 1	$B \cdot 10^{16}$	2,24	6	16,4	2,3	7.8	23
	p (мгц)	51	39	20,2	57	44	27
Klystron № 2	$B \cdot 10^{16}$	2,12	3,14	10,6	2,8	3,3	20
	p (мгц)	53	43	27,5	55	48	33
Theoretical Values	$B \cdot 10^{16}$	2,6	9	27	4.5	12	29
	p(Mc)	56	34	19	66	40	34

Table 2

		$\omega_0\tau_0=13{,}5\pi$	$\omega_0\tau_0>13{,}5\pi$	$\omega_0\tau_0<13{,}5\pi$	$\omega_0\tau_0=15{,}5\pi$	$\omega_0\tau_0>15{,}5\pi$	$\omega_0\tau_0<15{,}5\pi$
(1) Клистрон № 1	(2) $\tilde{w}_\nu$ (рад²·сек⁻¹)	1,06	1,75	2,8	2,4	4,8	7,4
	ΔF (гц)	0,085	0,14	0,22	0,191	0,38	0,58
	$\Delta\, 10^{-7}$ (рад·сек⁻¹)	—	2,2	−2,4	—	6,4	−3,5
	w_ν (рад²·сек⁻¹)	—	1,46	1,9	—	1,6	4,5
(3) Клистрон № 2	$\tilde{w}_\nu$	1,14	1,84	4,5	1,8	4,6	7,3
	ΔF	0,091	0,145	0,36	0,143	0,37	0,58
	$\Delta\, 10^{-7}$	—	3,2	−4,7	—	5,4	−3,3
	w_ν	—	1,53	2,2	—	1,9	5,2
(4) Теоретические значения	ΔF (гц)	0,087	0,09	0,173	0,133	0,135	0,29
	$\Delta\, 10^{-7}$	0	10	−13	0	15	−7
	w_ν	1,09	1,12	2,16	1,67	1,7	3,64

(1) Klystron No. 1

(2) w_ν (rad² - sec⁻¹)

F (cps)

10^{-7}(rad-sec⁻¹)

w_ν (rad² - sec⁻¹)

(3) Klystron No. 2

(4) Theoretical values F (cps)

for constant attention to the work.

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THE PROBLEM OF THE METHOD OF DETECTOR CHARACTERISTICS

[This is a translation of an article written by K. I. Kononenko in Radiofizika (Radiophysics) Vol. II, No. 6, 1959, pages 927-930.]

(Abstract) A further development is given in the paper of the method of detector characteristics. An equation is introduced for the detector characteristics which makes it possible to determine the potential of plasma, and the temperature and density of electrons in the plasma directly from experimental detector characteristics.

As was shown in [1], the detector characteristics make it possible to determine the basic parameters of gap discharge plasma--potential, temperature and density; moreover, one can determine the electron velocity distribution function from the experimental detector characteristic.

As is well known, the sonde characteristic of a negatively charged plane sonde can be represented in the following form:

$$I = \frac{e_0 N v}{4} e^{eV/kT} = I_0 e^{-eV/kT}, \qquad (1)$$

where I is the sonde current, e_0 is the electron charge, v is the mean velocity of the electrons, V is the potential of the sonde relative to the potential of the plasma, k is the Boltzmann constant, T is the temperature of the electrons, N is the density of the plasma.

If we limit ourselves to a consideration of quadratic detection, then we shall have the following expression for the value of the detector current:

$$\Delta I = \frac{\partial^2 I}{\partial V^2} \left(\frac{V_\sim}{2} \right)^2, \qquad (2)$$

where $V_\sim$ is the amplitude of the alternating detector voltage.

Determining the second derivative from (1) and substituting the resultant expression in (2), we obtain:

$$\Delta I = BNT^{3/2} e^{-eV/kT} V_\sim^2, \qquad (3)$$

where

$$B = e_0^3 / \sqrt{32 k^3 \pi m}. \qquad (4)$$

On the other hand, as Druyvesteyn has shown [2], the distribution function can be represented in the form:

$$\rho\left(\sqrt{2 e_0 V / m}\right) = \frac{4m}{e_0^2} V \frac{\partial^2 I}{\partial V^2}. \qquad (5)$$

Substituting here the value of the second derivative from (2), we obtain:

$$\rho = FV\Delta I, \qquad (6)$$

where the coefficient

$$F = 16m/e_0^2 V_{\sim}^2 \tag{7}$$

in the construction of the distribution function can be regarded as a constant, inasmuch as the amplitude of the detector voltage does not change taking down the detector characteristic. Substituting in (6) the value of the detector current ΔI in correspondence with Eq. (3), we obtain the distribution function in explicit form:

$$\rho = \frac{e_0}{k}\sqrt{\frac{8m}{\pi k}}\, N T^{-3/2}\, V e^{-e_0 V/kT}. \tag{8}$$

This function passes through a maximum at the point where

$$e_0 V_{\rho_M} / kT = 1 . \tag{9}$$

It is then easy to determine the temperature of the electrons

$$T = (e_0/k)\, V_{\rho_M} , \tag{10}$$

where V_{ρ_M} is the potential corresponding to the maximum of the distribution function.

To obtain the density of the plasma, we compute the maximum of the distribution function (8). Making use of the condition (9) and eliminating the temperature, we find that

$$\rho_M = \sqrt{\frac{8m}{\pi e_0}}\, \frac{N}{e\sqrt{V_{\rho_M}}} .$$

Solving the equatinn thus obtained for N, we obtain a formula for the determination of the plasma density:

$$N = D\rho_M \sqrt{V_{\rho_M}} ; \qquad D = e\sqrt{\frac{\pi e_0}{8m}} = \text{const} . \tag{11}$$

In the resultant expression we can introduce the temperature in place of the potential. In this case, the equation for the density of the plasma takes the following form:

$$N = D'\rho_M \sqrt{T}\ ; \quad D' = e\sqrt{\frac{\pi k}{8m}} = \text{const}. \tag{12}$$

Thus, by taking down the detector characteristic and constructing the distribution function in accord with (6), we can compute the temperature with the aid of (10), while with the help of (11) or (12) we can determine the density of the plasma. As was shown in [3, 4], the method of detector characteristics is most accurate in comparison with all other sonding methods. Its value lies in the fact that it makes possible an experimental determination of the distribution function.

In the method set forth for the determination of the parameters of the plasma, it is necessary initially to construct the distribution function from the detector characteristic, and only then to determine the parameters of the plasma from the maximum of this function.

We shall now show that the parameters can be obtained directly from the detector characteristic.

According to [5], the detector characteristic is satisfactorily approximated by the function

$$\Delta I = A' V^{3/2} e^{\,\varepsilon_0 V/kT}, \tag{13}$$

where A' is an experimental constant. We write down this formula

in a somewhat different form, namely:

$$\Delta I = A V^{1/2} e^{-e_0 V/kT} V_{\sim}^2, \qquad (14)$$

where A is a certain new constant, $V_{\sim}$ is the amplitude of the alternating detected voltage. The latter formula corresponds to the case of square law detection.

The function (14) passes through a maximum at the point where the condition

$$2e_0 V = kT. \qquad (15)$$

is satisfied. From this condition on the potential $V = V_M$, which corresponds to a maximum of the detector current function, one can easily obtain the temperature T. Equating (9) and (15), we obtain

$$V_{\rho_M} = 2V_M. \qquad (16)$$

For the determination of the plasma density, we compute the value of the detector current from Eqs. (3) and (14) at the maximum of the distribution function:

$$\Delta I_{\rho_M} = BNT^{3/2} e^{-1} V_{\sim}^2;$$

$$\Delta I_{\rho_M} = A V_{\rho_M}^{1/2} e^{-1} V_{\sim}^2. \qquad (17)$$

Comparing the right hand sides of these equations, we obtain:

$$BNT^{3/2} - A V_{\rho_M}^{1/2}. \qquad (18)$$

Eliminating the temperature by means of (15) and transforming from V_{ρ_M} to V_M by means of Eq. (16), we obtain the following expression for the plasma density:

$$N = DV_M^2. \qquad (19)$$

where D is a constant associated with the constant A in Eq. (14)

through the relation

$$D = 16\sqrt{2\pi m e_0^{-3}}A \; . \tag{20}$$

We now show that the experimental constant A can generally be eliminated from discussion. For this purpose we make use of the relation (18). Eliminating V_{ρ_M} by means of (9), and solving relative to A, we find that

$$A = B\left(\frac{e_0}{k}\right)^{1/2} NT^{-2} . \tag{21}$$

Substituting the latter value of A in (14) and taking (4) into account, we obtain:

$$\Delta I \quad \frac{e_0^2}{k^2\sqrt{32\pi m}} NT^{-2} V^{1/2} e^{e_0 V kT} V^2 . \tag{22}$$

This expression represents the curve of the detector current in which no experimental constants appear. Thus we have obtained an analytic expression for the detector characteristic of a function of the fundamental parameters of the plasma: potential, temperature and plasma density, and also of the value of the amplitude of the detected voltage.

The plasma potential enters in (23) through the expression for V by means of the formula

$$V = V_s - V_{pl} \tag{23}$$

where V_s is the potential of the sonde relative to the anode, the cathode or any other reference electrode, V_{pl} is the plasma potential. The plasma potential is determined by the intersection of the detector characteristic with the abscissa.

Making use of Eq. (23), we now obtain the connection between the values of the detector current at the maximum of the distribution function and at the maximum of the detector current function. Taking it into account that at these points

$$\Delta I_M = ANT^{-2} V_M^{1/2} e^{-1/2} V_\sim^2 ;$$

$$\Delta I_{\rho_M} = ANT^{-2} V_{\rho_M}^{1/2} e^{-1} V_\sim^2 ,$$

we obtain:

$$\Delta I_M / \Delta I_{\rho_M} = \sqrt{e V_M / V_{\rho_M}} = \sqrt{e/2} \tag{24}$$

[see (16)].

We shall now show that the plasma density can be obtained directly from the detector characteristic without having recourse to calculation of the distribution function, and without making use of any experimental constant. For this purpose we substitute the value of ρ_M in Eq. (11), making use of (6). As a result we obtain:

$$N = D\rho_M \sqrt{V_{\rho_M}} = DFV_{\rho_M} \Delta I_{\rho_M} \sqrt{V_{\rho_M}} .$$

Taking (17) and (24) into account, and transforming from V_{ρ_M} and ΔI_{ρ_M} to the values of these quantities at the maximum of the detector current function, we have:

$$n \simeq \frac{4DF}{\sqrt{e}} \Delta I_M V_M^{3/2} .$$

Substituting the values of the constant coefficients, we finally obtain the result that

$$N = M \Delta I_M V_M^{3/2} ; \quad M = 16 V_\sim^{-2} \sqrt{2\pi m e e_0^{-3}} . \tag{25}$$

The quantity M can be regarded as a constant since the amplitude of the alternating detected voltage $V_{\sim}$ remains unchanged in taking down the detector characteristic. For other measurements, in which $V_{\sim}$ will take on different values, the constant M will also change.

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A STUDY BY BENDIXSON'S METHOD OF THE TOPOLOGICAL STRUCTURE OF THE LOCATION OF TRAJECTORIES IN THE VICINITY OF A SINGULAR POINT OF A DYNAMICAL SYSTEM

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(Abstract) All possible topological structures of singular points of a dynamical system containing linear terms in right parts are established in the paper by means of the method of Bendixson.

The dynamical system

$$dx/dt = P(x, y); \quad dy/dt = Q(x, y), \tag{1}$$

is considered, the right hand sides of which are analytic functions in a certain region about the origin. It is assumed that the origin O(0, 0) is an isolated singularity of the system (1), for which the following conditions are satisfied

$$\Delta(0, 0) = 0;$$
$$\delta(0, 0) = 0;$$
$$|P'_x(0, 0)| + |P'_y(0, 0)| + |Q'_x(0, 0)| + |Q'_y(0, 0)| \neq 0.$$

Here
$$\Delta(x, y) = \begin{vmatrix} P'_x(x, y) & P'_y(x, y) \\ Q'_x(x, y) & Q'_y(x, y) \end{vmatrix};$$
$$\delta(x, y) = P'_x(x, y) + Q'_y(x, y).$$

It follows from the results of Bendixson [1] that the topological structure of the distribution of trajectories in a sufficiently small region around a singular point of the system (1) which is not at the center is completely determined by the number of closed nodes, open nodes and saddle regions which abut the given singular point and by the mutual arrangement of these regions. (We shall denote these numbers by the letters N_f, N and C, respectively. Here the open nodal regions which accompany the closed nodal regions do not enter into the number N.) Let us make this concept precise. Let the trajectory L tend to the singular point O both at $t \to +\infty$, and at $t \to -\infty$, and let there be no singular point of the system (1) inside the simple closed curve C_L, consisting of the trajectory L and the point O. We denote the region bounded by the curve C_L by σ_δ. Then any trajectory passing into the region also approaches the point O for $t \to +\infty$ and for $t \to -\infty$ (see [1]), forming, together with the point O, a simple closed curve. If one region of each two regions bounded by such curves lies inside the

other, then we call the region σ_δ the <u>closed nodal region</u> of the singular point O. Two closed nodal regions of the point O are different if they are located one inside the other.

Let us consider further the circle C_r of radius r with center at the singular point O. We assume that 1) inside C_r there are no other singular points of the system (1) except the point O, 2) outside of C_r there are known to lie points if only of a single half-trajectory, tending to the point O, 3) outside C_r there are known to lie points of each of the closed regions of the point O. Let L_1 and L_2 be two half-trajectories tending to the singular point O and having common points with the circle C_r. Let M_1 and M_2 be the last common points of these half-trajectories with the circle C_r. We denote by σ the region whose boundary consists of the part M_1O of the half-trajectory L_1, the part M_2O of the half-trajectory L_2, the point O, and that of the part of the circle C_r within M_1, M_2, on which motion in the direction from the point M_1 to the point M_2 is positive (counter-clockwise).

If there exists a circle C_ρ with center at the point O of radius $\rho \leq r$, such that all trajectories passing through points of the region σ lying inside the circle C_ρ for $t \to +\infty$ $(t \to -\infty)$ not emanating from σ approach O, and with decrease (increase) of t emanate from σ, then the region between the half-trajectories L_1 and L_2 is called an <u>open nodal region</u> of the point O. If there is no other nodal region here containing the one considered, and penetrating to the

boundary, of at least one of the half-trajectories L_1 and L_2, then the region between the half-trajectories L_1 and L_2 is known as a complete open nodal region. Above, by the number N is meant the number of complete open nodal regions. Furthermore, in speaking of complete open nodal regions, we will omit the word "complete."

If all the trajectories passing inside the region σ, both for increase and for decrease in t, emanate from σ, then the region between the half-trajectories L_1 and L_2 is known as a saddle region of the point O.

For the singular points considered by us, the numbers N_f, N and C are such that the topological structure of the singular point is determined uniquely by them. In this case in addition to the known singular points (node, focus, saddle and saddle-node), the following two additional types of points are encountered.

1. A singular point which is abutted by only two saddle regions; for this point $N_f = 0$, $N = 0$, $C = 2$. We shall call such a point a degenerate singular point.

2. The singular point which is abutted by one closed nodal region and one saddle region; for this point $N_f = 1$, $N = 0$, $C = 0$. We shall call such a point a point with a closed nodal region.

We shall say that the half-trajectory $x = x(t)$, $y = y(t)$ reaches the singular point O(0, 0) for $t \to +\infty$ in the direction $y = kx$ (in the direction $x = 0$), if $x(t) \to 0$, $y(t) \to 0$ for $t \to +\infty$ and $\lim_{t \to +\infty} y(t)/x(t) = k$ ($\lim_{t \to +\infty} x(t)/y(t) = 0$). In a similar way, we shall speak of the

half-trajectory touching the point $O(0, 0)$ for $t \to -\infty$, or simply of the half-trajectory reaching the point $O(0, 0)$ if we are indifferent as to whether t tends to $+\infty$ or to $-\infty$.

For the system (1) let $O(0, 0)$ be a singularity and

$$P(x, y) = P_m(x, y) + P_{m+1}(x, y) + \dots,$$
$$Q(x, y) = Q_m(x, y) + Q_{m+1}(x, y) + \dots$$

be the expansions of the functions $P(x, y)$ and $Q(x, y)$ in powers of x and y, in which P_k and Q_k are homogeneous polynomials of kth degree and m is the smallest number for which P_m and Q_m are simultaneously not identically equal to zero.

Bendixson's Theorem ([1], page 34). Let the system (1) have an isolated singularity at the origin $O(0, 0)$. Then the half-trajectory of the system (1), which approaches O for $t \to \pm\infty$, will either be a spiral or will pass through the point O in a direction satisfying the equation $xQ_m - yP_m = O$.*

*The direction $y = kx$ for $x = 0$ satisfies the equation $xQ_m - yP_m = 0$ if the expression $y = kx$ for x is one of the linear factors of the homogeneous polynomial $xQ_m - yP_m$.

We note that if any one of the half-trajectories passes through the singularity O, then all the trajectories approaching O enter it. If there are no trajectories entering O, then the singularity O is a focus or center.

In the investigation of the system (1), Bendixson applied

bilinear transformations of the coordinates, in particular the transformation of the form:

$$x = x, \quad y = (\eta + k)\, x. \tag{2}$$

For k = 0, we obtain the transformation

$$x = x, \quad y = \eta x, \tag{3}$$

the properties of which are given below.

a) The transformations(3) unambiguously and continuously (topologically) map into one another the regions of the planes (x, y) and (x, η) which are obtained if one separates the points of the straight lines x = 0 from both planes. In this case the points of the first, second, third and fourth quadrants of one plane transform into the points of the first, third, second and fourth quadrants, respectively, of the other plane. All the points of the axis x = 0 of the plane (x, η) transform into the point O(0, 0) of the plane (x, y). At the points of the axis x = 0 of the plane (x, y) the transformation is not defined.

b) We consider the δ-vicinity U_δ of the point O(0, 0) of the plane (x, y). This region is divided by the axis x = 0 into two regions. In the transformation (3), each of these regions transforms into a region of the plane (x, η) which represents a band located on one side of the axis x = 0 and bounded by this axis and a line which approaches it asymptotically. Thus the entire region U_δ, excepting the cut of the axis x = 0, is mapped into the region of the plane (x, η) located between two lines which asymptotically approach the axis x = 0,

excluding the axis $x = 0$ itself. We denote this region by the letter Γ (Fig. 1).

Let us formulate the following obvious confirmation, which lies at the basis of the Bendixson method and which is systematically employed for the proof of fundamental assumptions.

Let the system (1) have a singularity at the origin. Applying the transformation (2), we transform the system (1) to the form:

$$dx/dt = P(x, \eta); \quad d\eta/dt = \overline{Q}(x, \eta). \tag{4}$$

Further, let $x = x(t)$, $y = y(t)$ be the half-trajectories of the system (1) which enter the point $O(0, 0)$ in the direction $y = kx$. Then the corresponding half-trajectory of the system (4) approaches the point $\overline{O}(0, 0)$, i.e., it either enters the point $\overline{O}(0, 0)$ or is a spiral. Conversely, each half-trajectory of the system (4) which approaches the point $\overline{O}(0, 0)$ (excluding the half-trajectories which are part of the η axis if there are such), corresponds to a half-trajectory of the system (1) entering the point $O(0, 0)$ in the direction $y = kx$.

Let us consider the system

$$dx/dt = P^*(x, \eta); \quad d\eta/dt = Q^*(x, \eta), \tag{5}$$

which is obtained from the system (1) by the transformation (3). Let the system (1) have in the region U_δ (see Fig. 1) no other singularities than the point $O(0, 0)$. On the basis of the properties a) and b) of the transformation (3), we conclude that only the points of the axis $x = 0$ can be the singular points of the system (5) located in the region Γ (Fig. 1). It is evident that each half-trajectory of the

system (5) which approaches the singularity $O_1(0, k)$ (excluding the half-trajectories which are parts of the η axis) corresponds to a half-trajectory of the system (1) entering the point $O(0, 0)$ in the direction $y = kx$.

The system (1) considered in the present note reduces, by a linear non-singular transformation*, to the form:

$$d\bar{x}/dt = \bar{y} + \bar{P}_2(\bar{x}, \bar{y}); \quad d\bar{y}/dt = \bar{Q}_2(\bar{x}, \bar{y}). \tag{6}$$

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*Under the condition $P'_y(0, 0) \neq 0$ the transformation

$$x \quad kx \quad \bar{y}.$$

$$P'_y(0, 0)\, y \quad kQ'_y(0, 0)\, x \quad \sqrt{|P'_y(0, 0)|}\, y.$$

where
$$k = \sqrt{|P'_y(0, 0)|} - Q'_y(0, 0)$$

is such a transformation.

Where $\bar{P}_2(\bar{x}, \bar{y})$ and $\bar{Q}_2(\bar{x}, \bar{y})$ are analytic functions in the vicinity of the origin whose expansions in powers of $\bar{x}$ and $\bar{y}$ begin with terms not smaller than the second order. Furthermore, the system (6) is reduced by the transformation

$$x = \bar{x}; \quad \bar{y} = y_1 + \varphi(x),$$

where $\varphi(\bar{x})$ is the solution of the equation $\bar{y} + \bar{P}_2(\bar{x}, \bar{y}) = 0$, to the form

$$d\bar{x}/dt = y_1; \quad dy_1/dt = \bar{Q}_2^*(x, \bar{y}_1).$$

It is easy to see that the topological structure of the location of the trajectories in the vicinity of the singular point O(0,0) of the system (1) does not change for the transformations we have made (a node in this case remains a node and a focus remains a focus by virtue of the analytical character of the transformation). Thus we have the possibility of considering a system of the form

$$dx/dt = y; \quad dy/dt = Q_2(x, y). \tag{7}$$

in place of the system (1), where $Q_2(x, y)$ is a function which satisfies the same conditions as does $\bar{Q}_2(\bar{x}, \bar{y})$. It follows from the assumption on the isolated character of the singularity of the system (1) that $Q_2(x, 0) \not\equiv 0$.

We now consider a number of propositions which are used in the proof of singularity theorems. By the transformation (3) $x = x$, $y = \eta x$, the system (7) reduces to the form:

$$dx/dt = \eta x; \quad d\eta/dt = Q_2(x, \eta). \tag{8}$$

Since the only direction in which the half-trajectories of the system (7) can enter the point O(0,0) is the direction $y = 0$, then each of the semiaxes $x = 0$, $\eta > 0$ and $x = 0$, $\eta < 0$ is an entire trajectory of the system (8) (see pages 7, 8). Let U_δ be the vicinity of the point O(0,0), and Γ be the corresponding region of the plane (x, η) (see Fig. 1). We assume that there exist the half-trajectories $\bar{L}$ and $\bar{L}'$ of the system (8), which reach the point $\bar{O}(0, 0)$ for $x > 0$ and $x < 0$, respectively. Let $\bar{N}$ and $\bar{N}'$ be the last common points of the half-trajectories $\bar{L}$ and $\bar{L}'$ with boundaries of the region

(Fig. 2), while L and L' are the half-trajectories of the system (7) corresponding to the half-trajectories $\bar{L}$ and $\bar{L}'$ of the system (8). By virtue of the property a) of the transformation (2), and with the confirmation formulated on page 6, the half-trajectories L and L' enter the point O(0, 0) for $x > 0$ and $x < 0$, respectively, and the points N and N', which correspond to the points $\bar{N}$ and $\bar{N}'$, are the last common points with the boundaries of the region U_δ (see Fig. 2).

1. Let us consider the regions abutting the point $\bar{O}(0, 0)$ of the plane (x, η) between the half-trajectories $\bar{L}$ and the half-trajectory $x = 0$, $\eta \geq 0$, and between the half-trajectory $\bar{L}'$ and the half-trajectory $x = 0$, $\eta < 0$. We denote them by the letters Γ_1 and Γ_2 (Fig. 2).

1) If these regions are open nodal regions at the point $\bar{O}$ of the system (8), then the region between the half-trajectories L and L' of the system (7) is a closed nodal region of the point O(0, 0).

2) If the regions mentioned are saddle regions of the point $\bar{O}$, then the region between the half-trajectories L and L' is also a saddle region of the point O(0, 0).

3) If one of the regions is a saddle region of the point $\bar{O}$, while the other is an open nodal, then the region between the half-trajectories L and L' is an open nodal of the point O. We note that instead of the regions Γ_1 and Γ_2 we could have in entirely similar fashion considered the regions Γ_1^* and Γ_2^* (see Fig. 2).

We now consider the system

$$dx/dt = xX(x,y); \quad dy/dt = Y(x,y), \tag{9}$$

where $X(x,y)$ and $Y(x,y)$ are analytic functions in a certain region about the origin and $Y(0,0) = 0$. We shall assume that the half-axes $x = 0$, $y > 0$ and $x = 0$, $y < 0$ are the only half-trajectories entering the point $O(0,0)$ in the direction $x = 0$, and also that there exist two directions $y = k_1 x$ and $y = k_2 x$, in which the half-trajectories of the system (9) can enter the singular point $O(0,0)$. Then, applying the transformation $x = x$, $y = \eta x$ and, in accord with the method of Bendixson, dividing the right hand side by the common factor $x^k (k \geq 1)$ if there is such, we obtain the set

$$dx/dt = xX^*(x,\eta); \quad d\eta/dt = Y^*(x,\eta), \tag{10}$$

for which the η axis consists of the half-trajectories and the points $O_1(0,k_1)$ and $O_2(0,k_2)$ are its singular points (see [1], page 62).

II. We assume that the singularity $O_1(0,k_1)$ is a saddle of the system (10) and that $\delta \neq 0$ at the point $O_2(0,k_2)$. Then, depending on whether the singularity $O_2(0,k_2)$ of the system (10) is a node, a saddle or a saddle-node,* the topological structure of the singular point O of the system (9) is completely characterized by the following values of the numbers C, N and N_f (see page 2):

1) If $O_2(0,k_2)$ is a node, then $C = 2$, $N = 2$, $N_f = 0$;

2) If $O_2(0,k_2)$ is a saddle, then $C = 6$, $N = 0$, $N_f = 0$;

3) If $O_2(0,k_2)$ is a saddle-node, then $C = 4$, $N = 1$, $N_f = 0$.

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*There can be no other cases by virtue of the fact that $\delta \neq 0$.

The assumptions I and II are proved by direct consideration of the location of the half-trajectories in the region Γ and the forms of these trajectories in the region U_δ, which is obtained in the transformation (3). In the proof properties a) and b) of the transformation (3) are employed, and also Bendixson's theorem and the statement formulated on page 6 . Here we obtain for the singular point O(0,0) of the system (9) a definite distribution of the separatrices as a function of the numbers k_1 and k_2.

III. The system

$$dx/dt = xy; \quad dy/dt = -ky^2 + x^2 f(x,y) + xyf_1(x,y), \qquad (11)$$

where $f(x,y)$ and $f_1(x,y)$ are analytic functions in the vicinity of the point O(0,0) and $k > 0$, has two and only two half-trajectories, entering the point O in the direction $x = 0$, and these are the positive and negative semiaxes y.

This proposition is proved by application of the transformation $x = \xi y$, $y = y$ to the system (11). For the system obtained after transformation and division of the right sides by the common factor y, the origin of the coordinates is a saddle, whose separatrices are the positive and negative semiaxes $\xi = 0$ and $y = 0$. In the transition to the system of coordinates x, y, the half-trajectories $y = 0$, $\xi > 0$ and $y = 0$, $\xi < 0$ are reflected in a single point O(0,0). The half-trajectories $\xi = 0$, $y > 0$ and $\xi = 0$, $y < 0$ transform into the half-trajectories $x = 0$, $y > 0$ and $x = 0$, $y < 0$ of the system (11).

IV. The system

$$dx/dt = xy; \quad dy/dt = ax[1 + \phi(x)] + by^2 + xyf(x,y), \tag{12}$$

is given in which $f(x,y)$ and $\phi(x)$ are analytic functions in the vicinity of the point $O(0,0)$. $a \neq 0$, $b < 0$ and $\phi(0) = 0$. The singular point $O(0,0)$ of this system is a saddle, whose separatrices are the half-trajectories $x = 0$, $y > 0$ and $x = 0$, $y < 0$ and two half-trajectories entering the point O in the direction $x = 0$ and located in the first and fourth quadrants (if $a > 0$) and in the second and third quadrants (if $a < 0$).

Proof of this statement will be given in more detail. It follows from Bendixson's theorem that there exists a single direction in which the trajectories of the system (12) can enter the point $O(0,0)$ (the direction $x = 0$). Applying the transformation $x = \xi y$, $y = y$, we obtain:

$$\begin{aligned} d\xi/dt &= (1 - b)\xi y - a\xi^2 - \xi^2 a\phi(\xi y) - \xi^2 yf(\xi y, y); \\ dy/dt &= a\xi y + by^2 + a\xi y\phi(\xi y) + \xi y^2 f(\xi y, y). \end{aligned} \tag{13}$$

The direction of the trajectories of the system (13) entering the singular point $\bar{O}(0,0)$ is obtained by virtue of the same theorem of Bendixson from the equation $\xi y[(1 - 2b)y - 2a\xi] = 0$. Two and only two half-trajectories of the system (13) enter the point $\bar{O}(0,0)$ in the direction $\xi = 0$: $\xi = 0$, $y > 0$ and $\xi = 0$, $y < 0$. (This can be shown in the same way as was done above.) For investigation of the two other directions $y = 0$ and $y = 2a\xi/(1 - 2b)$ the transformation $y = \eta\xi$, $\xi = \xi$ is employed. After transformation and division of the right side of (13) by the common factor ξ we obtain:

$$d\xi/dt = -a\xi + (1 - b)\,\eta\xi - a\xi\,\phi(\eta\xi^2) - \eta\xi^2 f(\eta\xi^2, \eta\xi);$$
$$d\eta/dt = 2a\eta - (1 - 2b)\,\eta^2 + 2a\eta\,\phi(\eta\xi^2) + 2\eta^2\xi f(\eta\xi^2, \eta\xi). \qquad (14)$$

It is easy to see that for the system (14) the η axis consists of the half-trajectories and the points $O_1(0, 0)$, $O_2(0, \frac{2a}{1 - 2b})$ are singularities. Thus one can apply the proposition II to the system (13). By direct calculation we obtain:

$$\Delta\,(0,\ 0) \qquad 2a^2 \quad 0; \quad \Delta\left(0, \frac{2a}{1-2b}\right) = \frac{2a^2}{1-2b} < 0.$$

Therefore both singularities O_1 and O_2 are saddles of the system (14), and by virtue of the proposition II (case 2) for the singularity $\bar{O}(0,0)$ of the system (13), $C = 6$, $N_f = 0$, $N = 0$, i.e., 6 half-trajectories enter the point $\bar{O}(0,0)$. Consequently, in addition to the half-trajectories which are the semiaxes $\xi = 0$ and $y = 0$, there are two half-trajectories entering the point $\bar{O}(0,0)$. We denote them by $\bar{L}_1$ and $\bar{L}_2$. It is not difficult to show that the half-trajectories $\bar{L}_1$ and $\bar{L}_2$ enter the point $\bar{O}$ in the first and third quadrants, respectively, if $a > 0$, and in the second and fourth quadrants if $a < 0$. Now, transforming to the system (12), we conclude that in addition to the half-trajectories $x = 0$, $y > 0$ and $x = 0$, $y < 0$, there are two and only two half-trajectories entering the point $O(0,0)$. We denote them by L_1 and L_2. They are images of the half-trajectories $\bar{L}_1$ and $\bar{L}_2$ obtained in the transformation $x = \xi y$, $y = y$. From the properties of this transformation, which are analogous to the properties a) and b) of the transformation (3), and from the

location of the half-trajectories $\bar{L}_1$ and $\bar{L}_2$ shown above, it follows that the half-trajectories L_1 and L_2 enter the point O in the first and fourth quadrants, respectively, if $a > 0$, and in the second and third quadrants if $a < 0$. Thus the point O(0, 0) of the system (12) is a saddle with the location of the separatrices as shown.

We now proceed to a consideration of the system (7) which can be written in the following form:

$$dx/dt = y;$$

$$dy/dt = a_\lambda x^k [1 + h(x)] + b_n x^n y [1 + g(x)] + y^2 f(x, y) \qquad (15)$$

($h(x)$, $g(x)$ and $f(x, y)$ are analytic functions in the vicinity of the origin, $h(0) = g(0) = 0$, $k \geq 2$, $n \geq 1$ and $a_k \neq 0$ by virtue of the properties of the function $Q_2(x, y)$). All possible topological structures which the singular point O(0, 0) of the system (7) can have are established in theorems 1 and 2.*

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*Effectually the same results were obtained independently in references [2] and [3]. in the method applied by Frommer.

Theorem 1. Let $k = 2m$ in system (15). Then the singularity O(0, 0) of the system (15) is a degenerate singular point (see page 4), if $b_n = 0$ or $b_n \neq 0$ and $n \geq m$ (Fig. 4a), and the singular point O(0, 0) is a saddle-node if $b_n \neq 0$ and $n < m$ (Fig. 3).

We note that Fig. 3a was completed under the conditions $b_n > 0$, $a_{2m} \leq 0$. Figure 3b corresponds to the conditions $bn \leq 0$,

$a_{2m} > 0$ and n an odd number or $b_n > 0$, $a_{2m} > 0$, n an even number, Fig. 3c corresponds to the conditions $b_n < 0$, $a_{2m} > 0$, n an even number, or $b_n > 0$, $a_{2m} > 0$, n an odd number. Figure 3d is completed under the conditions $b_n < 0$, $a_{2m} < 0$; Fig. 4a corresponds to the conditions $a_{2m} > 0$; in the case $a_{2m} < 0$, the location of the integral curves is obtained by a symmetric mapping relative to the y axis.

Proof. For the system (15) the only direction in which the trajectories can enter the point O(0,0) is the direction y = 0. Applying the transformation x = x, $y = \eta_1 x$, we reduce the system (15) to the form:

$$dx\ dt = x\eta_1;$$
$$\frac{d\eta_1}{dt} = -\eta_1^2 + a_{2m}x^{2m-1}\left[1 + h(x)\right] + b_n x^n \eta_1 \left[1 + g(x)\right] + \eta_1^2 x f_1(x, \eta_1). \tag{16}$$

under the conditions of Theorem 1.

For m = 1, $n \geq 1$, it follows from proposition IV that the singular point $O_1(0,0)$ of the system (16) is a saddle, for which the separatrices are the half-trajectories y = 0, $\eta_1 > 0$ and x = 0, $\eta_1 < 0$. The location of two other separatrices is determined by the sign of a_{2m}. In the transition to the system of coordinates x, y, the trajectories which are the semiaxes x = 0 are mapped into the single point O(0,0). Thus only two half-trajectories of the system (15) exist which enter the point O(0,0), and consequently the point O(0,0) is a degenerate singular point. In the case $a_{2m} > 0$, the location of

the trajectories is shown in Fig. 4a.

For $m \geq 1$, we carry out the system of transformation:

$$x = x,\ \eta_1 = \eta_2 x;\quad x = x,\ \eta_2 = \eta_3 x;\quad \ldots;\quad x = x,\ \eta_{i-1} = \eta_i x,$$

as a result of which the system (16) takes the form:

$$\begin{aligned} dx/dt &= \eta_i x; \\ d\eta_i/dt &= -i\eta_i^2 + a_{2m}x^{2m-2i+1}[1 + h(x)] + \\ &\quad + b_n x^{n-i+1}\eta_i[1 + g(x)] + \eta_i^2 x f_i(x, \eta_i). \end{aligned} \tag{17}$$

(This can be easily established by the method of induction).

We consider separately two cases.

1. Let $b_n = 0$, or $b_n \neq 0$ but $n \geq m > 1$. We write down the system (17) for $i = m$ in the following form:

$$\begin{aligned} dx/dt &= \eta_m x; \\ d\eta_m/dt &= -m\eta_m^2 + a_{2m}x[1 + h(x)] + \eta_m x f_m(x, \eta_m). \end{aligned} \tag{18}$$

For this system the point $\bar{O}_m(0,0)$ is a saddle by virtue of proposition IV. Let us consider the system (17), assuming that $1 \leq i \leq m - 1$. In this case $2m - 2i + 1 \geq 3$ and $n - i + 1 \geq 2$; consequently, for each number i the direction in which the trajectories of the system (17) can approach the point $\bar{O}_i(0,0)$ of the plane(x, η_i) are determined by the equation $x\eta_i^2 = 0$. Here two and only two half-trajectories (namely, the two semiaxes $x = 0$) enter the point $\bar{O}_i$ in the direction $x = 0$, by virtue of proposition III. The remaining half-trajectories of the system (17) $(i = 1, 2, \ldots, m - 1)$ entering the point $\bar{O}_i$(if there are such) enter this point in the direction $\eta_i = 0$.

To each half-trajectory there corresponds a half-trajectory entering the point $\bar{O}_{i+1}$ by virtue of the properties of the transformation $\eta_i = \eta_{i+1}x$, $x = x$. This half-trajectory belongs to the system (17) and corresponds to the value $i = i + 1$. This correspondence is single-valued if one does not take into account the half-trajectories $x = 0$, $\eta_i > 0$ and $x = 0$, $\eta_i < 0$. For the system (18), as was demonstrated in proposition IV, except for the half-trajectories $x = 0$, $\eta_m > 0$ and $x = 0$, $\eta_m < 0$, there exist two and only two half-trajectories entering the point $\bar{O}_m(0, 0)$. Then, for each of the systems (17) (for arbitrary $i < m$), including the system (16), there exist two and only two half-trajectories entering the point $\bar{O}_i$ (except for the half-trajectories $x = 0$, $\eta_i > 0$ and $x = 0$, $\eta_i < 0$). Consequently, the point $\bar{O}_1(0, 0)$ is a saddle point for the system (16), while the location of the separatrices will be the same as for the saddle point $\bar{O}_m(0, 0)$ by virtue of the properties of the transformation $\eta_i = \eta_{i+1}x$, $x = x$ $(i = 1, 2, \ldots, m - 1)$; the point is determined by the sign of the number a_{2m}. It therefore follows from the same considerations as the case $m = 1$ that the point $O(0, 0)$ of the system (15) is a degenerate singularity.

2. Now let $b_n \neq 0$ and $m > n \geq 1$. For $i = n$ the system (17) has the form:

$$
\begin{aligned}
&dx/dt = \eta_n x; \\
&d\eta_n/dt = a_{2m}x^{2m-2n+1}\,[1 + h(x)] - n\eta_n^2 + \\
&\quad + b_n x \eta_n\,[1 + g(x)] + \eta_n^2 x f_n(x, \eta_n).
\end{aligned}
\tag{19}
$$

Here $2m - 2n + 1 \geq 3$, so that the direction in which the trajectories can enter the point $\bar{O}_n(0,0)$ are determined by the equation

$$x\eta_n[(n+1)\eta_n - b_n x] = 0. \tag{20}$$

In the case considered two and only two trajectories ($x = 0$, $\eta_n > 0$ and $x = 0$, $\eta_n < 0$) enter the point $\bar{O}_n$ in the direction $x = 0$ (proposition III). It follows from Eq. (20) that there also exist two directions $\eta_n = 0$ and $\eta_n = b_n(n+1)^{-1}x$ in which the trajectories of the system (19) can enter the point $\bar{O}_n$. Applying the transformation $x = x$, $\eta_n = \eta_{n+1}x$, we investigate the demonstrated directions. The system (19) in this case takes the form:

$$\begin{aligned} dx/dt &= \eta_{n+1}x; \\ d\eta_{n+1}/dt &= -(n+1)\eta_{n+1}^2 + a_{2m}x^{2m-2n-1}[1 + h(x)] + \\ &+ b_n\eta_{n+1}[1 + g(x)] + \eta_{n+1}^2 x f_{n+1}(x, \eta_{n+1}). \end{aligned} \tag{21}$$

It is easy to see that for the system (21) the η_{n+1} axis is composed of half-trajectories and the points $O_1(0, \frac{b_n}{n+1})$, $O_2(0,0)$ are singular. Thus one can apply proposition III to the system (19). By direct calculation we obtain:

$$\Delta\left(0, \frac{b_n}{n+1}\right) = -\frac{b_n^2}{n+1} < 0,$$

i.e., the point $O_1(0, \frac{b_n}{n+1})$ is the saddle point of the system (21). Furthermore, we have for the point $O_2(0,0)$: $\Delta(0,0) = 0$ and $\delta(0,0) = b_n \neq 0$.

Such singular points were considered in [1]. They can be one

of the following three forms: nodal, saddle or saddle-nodal. The exact form taken in the given case can be determined as was done in [4] (pages 37-39). Actually, in the given case, the functions $\phi(x)$ and $\xi(x)$ have the form:

$$\varphi(x) = -\frac{a_{2m}}{b_n} x^{2m-2n-1} + \ldots \quad \text{и} \quad \psi(x) = -\frac{a_{2m}}{b_n^2} x^{2m-2n} + \ldots .$$

Thus the point under investigation is an even pole and by virtue of lemma 3 and theorem 2 of Par. 2 of the paper [4] it is saddle-nodal, the open nodal region of which is located to the right of the η_{n+1} axis if $a_{2m} < 0$, and to the left of this axis if $a_{2m} > 0$. Then, in accord with the proposition II (case 3), for the singular point of point $\bar{O}_n$ of the system (19): $C = 4$, $N = 1$, $N_f = 0$.

We note that the sign of the number b_n demonstrates whether the rough saddle O_1 of the system (21) is located above the saddle node O_2 or below, while the sign of the number a_{2m}, as was shown above, determines the location of the open nodal region of the saddle-node O_2 to the right or to the left of the η_{n+1} axis. Taking this into account, we obtain the definite location of the trajectories of the system (19) in the vicinity of the point $\bar{O}_n$ for different signs of the number a_{2m} and b_n.

The transition from the system (19) to the system (16) is the same as in the case $n \geq m$. However, because of the peculiarities of the location of the trajectories of the system (19) and the properties of the transformation $\eta_i = \eta_{i+1}x$, $x = x$ $(i = 1, 2, \ldots, n-1)$ the

case here when n is an odd number differs from the case when n is even. It is easy to see that for n odd, the location of the trajectories in the vicinity of the singularity $\bar{O}_1$ of the system (16) is the same as for the point $\bar{O}_n$ of the system (19). For n even, the location of the trajectories in the vicinity of the point $\bar{O}_1$ is obtained if we make a symmetric mapping of the half-plane (x, η_1) $(x < 0)$ relative to the x axis for the case corresponding to odd n and corresponding to the signs of the numbers a_{2m} and b_n. On the basis of proposition 1, we now conclude that the point O(0,0) of the system (15) is saddle-nodal, since for this point, $C = 2$, $N = 1$, $N_f = 0$. It is not difficult to see that for the corresponding signs of the numbers a_{2m} and b_n, the trajectories in the vicinity of the point O are located the same as is shown in Fig. 3. Thus theorem 1 is proved completely.

Theorem 2. Let $k = 2m + 1$ and $b_n^2 + 4(m + 1)a_{2m+1} = \lambda$ in the system (15). Then, if $a_{2m+1} > 0$, the singular point O(0,0) of the system (15) is a saddle point (Fig. 4b). If $a_{2m+1} < 0$, the singularity O(0,0) is 1) a focus or center for $b_n = 0$, while for $b_n \neq 0$ and $n > m$ or for $b_n \neq 0$, $n = m$ and $\lambda < 0$; 2) nodal if n is an even number and in this case $n < m$ or $n = m$ and $\lambda \geq 0$ (Fig. 4c); 3) a point with a closed nodal region (see page 3) if n is an odd number and $n < m$ or $n = m$ and $\lambda \geq 0$ (Fig. 4d). We note that Fig. 4c and 4d are completed under the condition $b_n > 0$; in the case $b_n < 0$, the location of the integral curves is obtained by a symmetric mapping relative to the x axis.

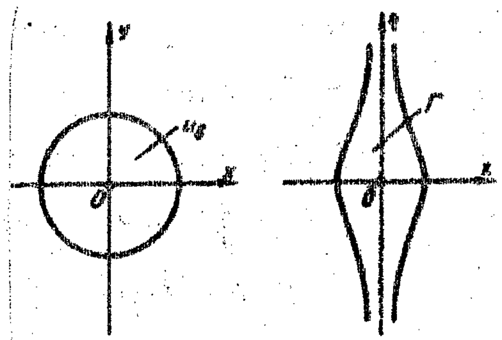


Fig. 1.

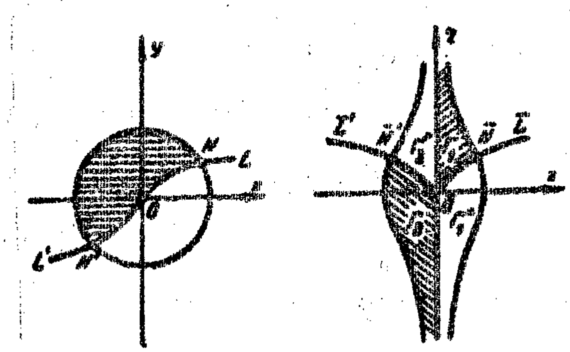
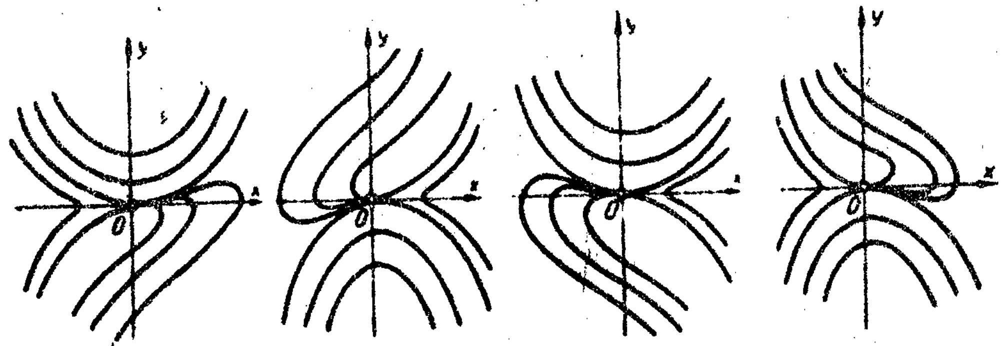
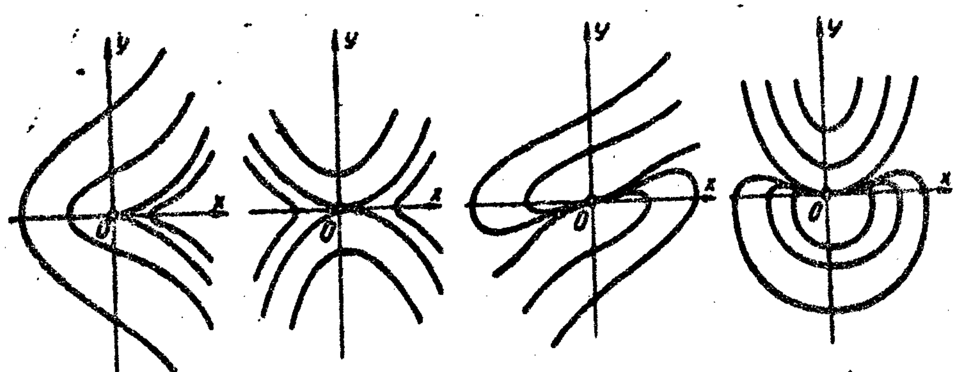


Fig. 2.



a) b) c) d)

Fig. 3.



a) b) c) d)

Fig. 4.

Proof of theorem 2 is carried out in a fashion similar to proof of theorem 1 and we shall omit it.

For determination of the type of the singularity of the system (6) there is no necessity of transforming from the system (6) to the system (15). The numbers k, n, a_k and b_n which characterize the types of singularities are determined by the early terms of the expansion in powers of x of the functions

$$\psi(\bar{x}) = \bar{Q}_2(\bar{x}, \varphi(\bar{x})) = a_k \bar{x}^k + \ldots;$$
$$\delta(\bar{x}, \varphi(\bar{x})) = \bar{P}'_{2x}(\bar{x}, \varphi(\bar{x})) + \bar{Q}_{2y}(\bar{x}, \varphi(\bar{x})) = b_n \bar{x}^n + \ldots,$$

where $\phi(\bar{x})$ is the solution of the equation $\bar{y} + \bar{Q}_2(\bar{x}, \bar{y}) = 0$ (see [4], page 55).

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THE POINT TRANSFORMATION OF A STRAIGHT LINE INTO A STRAIGHT LINE

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(Abstract) A further study is presented of the point transformation of a straight line into a straight line in the light of problems arising in the investigation of concrete systems. A number of theorems are established on the regions of attraction of fixed points, the bifurcations of multiple fixed points of a sufficiently smooth transformation are studied and the bifurcations of fixed points of a discontinuous piecewise linear transformation are taken into consideration.

As is well known, the study of many vibrational systems met with in radiophysics and automatic regulation reduce to the investigation of a point transformation of a straight line into a straight line [1-5].

The method of point transformations was developed and first

applied to problems of regulation theory by A. A. Andronov and his co-workers [1,2,6,7]. The systematic exposition of certain general questions of the method of point transformation is given in [8-10]. The iteration method developed previously [11] was applied to the study of vibrational systems in [12, 13]. The principal results of this method were used in the method of point transformation.

Let a transformation of a straight line into a straight line be given in a certain region:

$$\bar{x} = f(x). \tag{1}$$

To each point $M_0(x_0)$ of the straight line from the region of definition of the transformation (1), which we denote by T, there corresponds a succession of iterations $M_1 = TM_0$, $M_2 = T^2M_0$, ... The point M^* is called a fixed point (f.p.) of the transformation T if $TM^* = M^*$. The fixed point M^* is stable if any succession of iterations, beginning in the sufficiently small region around the point M^*, returns to M^*, i.e., if, for any suitably small $\varepsilon > 0$, there is found such a $\delta > 0$ that for any $x_0 \in (x^* - \quad, x^* + \quad)$ and arbitrary k the inequality $|x_k - x_k^*| < \varepsilon$. The fixed point M^* is unstable if for any suitably small $\varepsilon > 0$ there is found a diverging sequence of iterations, beginning in the δ-region about the point M^*. The necessary and sufficient condition for stability is the condition [11] $|df(x^*)/dx| < 1$, and for instability, $|df(x^*)/dx| > 1$.

The point $M_1^*(x_1^*)$, such that $x_1^* = T^n x_1^*$, but $x_1^* \neq T^m x_1^*$ for all $m < n$, is known as an n-fold fixed point of the transformation T.

It is stable if $|df^n(x_1^*)/dx| < 1$, where f^n is the nth iteration of the function f(x). The function $\lambda_i = df^n(x_i^*)/dx$ is known as the characteristic root of the fixed point M_i^*. Together with the point M_1^*, the points $M_{i+1}^* = T_i M_1^*$ $(i = 1, 2, \ldots, n - 1)$ are also n-fold fixed points [11]. The set of n-fold fixed points M_i^* $(i = 1, 2, \ldots, n)$ is called cyclic since they permute cyclically in the application to them of the transformation T and consequently the succession of iterations of the n-fold fixed points T is periodic. We note that the n-term cycle of the transformation T is stable if the component fixed points of the transformation T are stable; the n-fold fixed points of T which comprise the cycle have the same characteristic roots [9] and consequently they are all either stable or unstable.

The set of points of the straight line the iterations of which reduce to the fixed point M^* is known as the region of attraction of the fixed point M^*. The region of attraction of the n-fold cycle divides into a region of attraction of n-fold fixed points, entering into it by a relation to the transformation T^n.

It follows from what is given above that each cycle is as a whole determined by one of the n-fold fixed points belonging to it. Therefore everything that has been said about the existence of the region of definition and stability of the fixed points automatically applies to the corresponding cycle.

In the first part of the work, theorems are established on the maximum possible multiplicity of the fixed points of the transforma-

tion (1), and the known theorems on the regions of attraction of stable fixed points [14] are generalized for the case of piecewise continuous transformations.

In the second part of the work there is a detailed investigation (applicable to the transformation of a straight line into a straight line) of the bifurcation of multiple fixed points for transitions through the surface N_{+1}, N_{-1} [9].

As noted in [9], in addition to the bifurcations connected with the transition through N_{+1}, N_{-1}, N_{ϕ}, bifurcations are also possible that are brought about by the disruption of the continuity of the transformation. In the third part of the work, discontinuous piecewise linear transformations of straight lines into straight lines are considered for which the simple fixed point is absent because of its discontinuity. In this case a division of the space of the parameters is made into a region of existence of fixed points of various types. We note that the discontinuous point transformations have already been frequently investigated in connection with concrete cases of vibration theory and automatic regulation [2-5].

1. FIXED POINTS: THE EXISTENCE OF REGIONS OF ATTRACTION

We suppose a point transformation of a straight line into a straight line $\bar{x} = f(x)$, where $x \in (a, b)$.

Theorem 1. Let $f(x)$ be a continuous function increasing in the interval (a, b). Then the transformation T has only a simple fixed point of alternating stability.

We assume that T has an n-fold fixed point x_1^*. Then T has a cycle $x_1^*, \ldots, x_n^*$, where $x_1^* = f(x_n^*)$, $x_{i+1}^* = f(x_i^*)$ $(i = 1, 2, \ldots, n-1)$, and $x_i^* \neq x_j^*$ for $i \neq j$. For definiteness, let $x_2^* > x_1^*$; then, by virtue of the fact that f is increasing, $x_1^* < x_2^* < \ldots < x_n^* < x_1^* <$ $< \ldots$, whence $x_1^* < x_1^*$. The resulting contradiction proves the theorem.

Since f is a continuous function, then neighboring fixed points of T cannot have identical stability. Therefore, on the basis of theorem 1 of reference [14], it can be established that the interval (a, b) divides into unstable fixed points of T on the region of attraction of stable fixed points. The boundaries of the regions of attraction of each stable fixed point coincide with the unstable fixed points of T closest to it.

Theorem 2. If f(x) is a continuous function decreasing in the interval (a, b), then T can have fixed points of multiplicity not greater than two and not greater than one simple fixed point.

Actually, there cannot be two different simple fixed points x_1^* and x_2^* of the transformation T, since for x_1^* x_2^*, we come to a contradiction (by virtue of the fact that f is decreasing) because it follows from $f(x_1^*) < f(x_2^*)$ that $x_1^* < x_2^*$. The function $f[f(x)] = f^2(x)$ is obviously continuous and increasing (a, b). Therefore, on the basis of theorem 1, the transformation T^2 has only simple fixed points of alternating stability. Since the fixed points of T^2 are fixed points of T^{2n} [11] (and there are no other fixed points of T^{2n})

(on the basis of theorem 1) then there can be no transformation T of fixed points of odd multiplicity greater than unity.

One can generalize theorems 1 and 2 to a broader class of functions f(f), namely, to the class of functions almost everywhere continuous in the interval (a, b), which satisfy certain additional conditions which will be supplied below. From proof of theorem 3 (see below) we require the following verification.

Let $f(x)$ be a function which is almost everywhere continuous in the interval (a, b), having discontinuities of two types: at points a_i, the values of $f(a_i - 0)$ $f(a_i + 0)$ are simultaneously either larger or smaller than a_i, while at the points of discontinuity b_i, $f(b_i - 0) < b_i < f(b_i + 0)$. Moreover, let T have simple fixed points α_j. Then between any two points b_i there is at least one point α_j.

For proof of this, let us assume that there are no points α_j in the interval (b_{i-1}, b_i). Then, in the vicinity of b_{i-1}, the function $f(x) > x$, in the vicinity of b_i, $f(x) < x$, and in the interval (b_{i-1}, b_i), the difference $f(x) - x$ must change sign. There are no points α_j in the interval (b_{i-1}, b_i). Consequently, there must exist such a point γ that $f(\gamma - 0) > \gamma > f(\gamma + 0)$; however, by the condition on $f(x)$, there are no such discontinuities. Therefore, there must be at least one point α_j in (b_{i-1}, b_i).

Theorem 3. Let the points b_i, α_j divide the interval (a, b) into the intervals (σ_{k-1}, σ_k) (which we denote by δ_k). Then the transformation T can have only simple fixed points. Each stable

fixed point α_j has its region of attraction, which is bounded by the neighboring b_i or by unstable α_j.

The proof of theorem 3 is similar to what was given for theorem 1. There are neither α_j nor b_i in the interval δ_k. Consequently, in δ_k the function f(x) is only larger or only smaller than x, and T has only simple f.p. in the vicinity of σ_i, $f(x) > x$, if $x > \sigma_i$, and $f(x) < x$ if $x < \sigma_i$. As x departs from σ_i, the difference f(x) - x can change sign only continuously, which follows from the conditions laid on f(x), while, since in the range (σ_{i-1}, σ_i) the function $f(x) < x$, while in (σ_i, σ_{i+1}), the function $f(x) > x$, then $\sigma_{i-1}, \sigma_{i+1}$ is a stable point α_j.

Let σ_{i-1} and σ_i be stable α_j. Then $f(x) < x$ in the right semicircle σ_{i-1}, and $f(x) > x$ in the left semicircle σ_i. Since there are neither α_j nor β_i in δ_i, then f(x) is either always greater than x or always less than x in δ_i. Thus σ_{i-1}, σ_i cannot simultaneously be stable points α_j.

It follows from what was said above that 1) the stable fixed points α_j on the one hand, and the points b_i with unstable points α_j on the other, alternate; 2) if in δ_k the function $f(x) > x$ (or $f(x) < x$), then in δ_{k-1}, δ_{k+1}, $f(x) < x$ (f(x) x).

Let α_j be a stable f.p. of the transformation T which is a boundary point for the intervals δ_k, δ_{k+1}. The intervals δ_k and δ_{k+1} serve as a region of attraction of the points α_j. In δ_k there is the inequality $\alpha_j > f(x) > x$, in δ_{k+1} the relation $\alpha_j < f(x) < x$.

Consequently, the succession of iterations $\{x_n\}$ of an arbitrary point of the intervals δ_k, δ_{k+1} converges and for $n \to \infty$, $x_n \to a_j$.

Theorem 4 follows from theorem 3.

Theorem 4. If $f(x)$ is such that $f[f(x)]$ satisfies the conditions of theorem 3, then the transformation T can have fixed points of multiplicity not greater than two.

In particular, if $f(x)$ is a function almost always continuous in (a, b), which does not increase in the regions of discontinuity, where ~~then~~ at the point of discontinuity $f(a_{i-0}) > f(a_{i+0})$, then $f^2(x)$ satisfies the conditions of theorem 3.

2. BIFURCATION OF MULTIPLE FIXED POINTS OF THE TRANSFORMATION OF A STRAIGHT LINE INTO A STRAIGHT LINE FOR TRANSITIONS THROUGH THE BOUNDARY SURFACE N_{+1}, N_{-1}

Let the transformation T, which depends on the parameter μ and which consists of successive applications of the transformations.

$$Y = T_1 X = f_1(X, \mu); \tag{2.1}$$

$$Z = T_2 Y = f_2(Y, \mu), \tag{2.2}$$

have a binomial cycle M_1^*, M_2^* at $\mu = 0$ such that $M_2^* = T_1 M_1^*$, $M_1^* = T_2 M_2^*$. Here M_1^* is an f.p. of the transformation $T_2 T_1$, M_2^* is an f.p. of the transformation $T_1 T_2$. The characteristic roots of the points M_1^* and M_2^* are equal to one another [9].

Let us first consider bifurcation of the f.p. M_1^* and M_2^* in the case in which their characteristic roots are equal to +1. Let

$X_1, f_1(X_1, 0)$ be the coordinates of the point M_1^* and let the transformation T_1 have the form:

$$y = \beta_1 x + \ldots + \beta_{k_1} x^{k_1} + \ldots, \tag{2.3}$$

in the vicinity of the point $M_1[X_1, f_1(X_1, \mu)]$, and the transformation T_2 in the vicinity of the point $M_2 = T_1 M_1$,

$$z = \alpha_1 y + \ldots + \alpha_{k_2} y^{k_2} + \ldots, \tag{2.4}$$

where, for $\mu = 0$,

$$\alpha_2 = \alpha_3 = \ldots = \alpha_{k_2 - 1} = \beta_2 = \beta_3 = \ldots = \beta_{k_1 - 1} = 0 ;$$

$$\alpha_1 = a_1 \neq 0, \quad \beta_1 = \bar{\beta}_1 \neq 0, \quad \alpha_{k_2} \neq 0, \quad \beta_{k_1} \neq 0 .$$

Then the transformation $P = T_2 T_1$ in the vicinity of M_1 can be written in the form

$$\bar{x} = \gamma_0 + \gamma_1 x + \ldots + \gamma_m x^m + \ldots, \tag{2.5}$$

where, for $\mu = 0$,

$$\begin{gathered} \gamma_0 = \gamma_2 = \gamma_3 = \ldots = \gamma_{m-1} = 0; \quad \gamma_1 = 1; \quad \gamma_m \neq 0; \\ m = \min(k_1, k_2); \quad \gamma_0 = -X_1 + f_2 | f_1 (X_1, \mu), \mu |; \\ \gamma_n = \sum_{i=1}^{n} \alpha_i \left(\sum_{j_1 + \ldots + j_i = n} \beta_{j_1} \ldots \beta_{j_i} \right) \quad (n = 1, 2, \ldots, m). \end{gathered} \tag{2.6}$$

For $\bar{x} = x = x^*$, we obtain the equation for the coordinates of the f. p.:

$$\delta_0 + \delta_1 x^* + \ldots + \delta_{m-1} (x^*)^{m-1} + (x^*)^m = 0, \tag{2.7}$$

where

$$\begin{gathered} \delta_1 = (\gamma_1 - 1) [\gamma_m + x^* \gamma_{m+1} + \ldots]^{-1}, \quad \delta_i = \gamma_i [\gamma_m + x^* \gamma_{m+1} + \ldots]^{-1} \\ (i = 0, 2, 3, \ldots, m - 1). \end{gathered} \tag{2.8}$$

For $\mu = 0$, $\delta_i = 0$ $(i = 0, 1, \ldots, m - 1)$ and (2.7) has an m-fold null root. According to the theorem of Weierstrass [15], for a

sufficiently small μ, the equation (2.7) has not more than m roots which approach zero for $\mu \to 0$. The matrix of the quantities

$$a_{np} = \frac{\partial \gamma_n}{\partial \alpha_p}\bigg|_{\mu=0} = \begin{cases} 0, & p \neq n \\ \beta_1^n, & p = n \end{cases}, \quad b_{np} = \frac{\partial \gamma_n}{\partial \beta_p}\bigg|_{\mu=0} = \begin{cases} 0, & p \neq n \\ \tilde{\alpha}_1, & p = n \end{cases} \quad (n = 1, 2, \ldots, m)$$

has the rank m. Therefore, in the vicinity of the point $A(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)$ such that $\mu = 0$, the system (2.6) is solvable relative to α_i ($i = 1, \ldots, m$) in terms of γ_j and β_q. The vicinity of the point $(\delta_0, \ldots, \delta_{m-1})_{\mu=0}$ can be divided into the region of the existence of identical collections of real roots of the equation (2.7), i.e., into the region of existence of identical collections of f.p. P. As is well known, the real roots of (2.7) can appear and disappear only in pairs. If (2.7) has the roots

$$x_{2j-1} = \eta_j + i\zeta_j, \quad x_{2j} = \eta_j - i\zeta_j \quad (j = 1, 2, \ldots, p, \ 2p \leqslant m);$$
$$x_{2p+l} = \xi_l \quad (l = 1, 2, \ldots, m - 2p),$$

then the coefficients δ_q are polynomials in ξ, η, ζ:

$$\delta_q = P_q(\xi_l, \eta_j, \zeta_j). \tag{2.9}$$

and in the vicinity of the point Δ, the region Ω_p of the existence of two p complex roots is divided from the region Ω_{p-1} of the existence of ~~two~~ 2p-2 complex roots by the surface Γ_p. The equation of this surface is obtained in parametric form from (2.9) where one of the quantities ζ_γ is equal to zero.

It is not difficult to note that all the regions Ω_p ($p = 0, 1, \ldots, p_0$)

where p_0 is the nearest integer less than or equal to 0.5m, but the line on which all the roots of Eq. (2.7) are equal. Therefore, in the bifurcation, the complicated fixed point can be broken up into $m - 2p$ simple f.p. $(p = 0, 1, \ldots, p_0)$.

Now let the point $H(\delta_0, \delta_1, \ldots, \delta_{m-1})$, in motion along a certain curve in the vicinity of Δ, pass through all Q_p beginning with $p = p_0$. In the region Q_{p_0} there are no fixed points for m even, for m odd there is one fixed point. In the motion of the point H intersects the surface γ_p, as a result of which two fixed points are produced each time, one stable and the other unstable [9]. Thus, for even m, the number of stable fixed points is equal to the number of unstable, for odd m it is one greater than the points of stability producing fixed points. For sufficiently small μ, $\gamma_1 = \alpha_1\beta_1$ and is close to unity; in a sufficiently small vicinity of the point M^*, the expression (2.5) is an increasing function of x. According to theorem 1 the fixed points of different stability alternate.

Thus, the following theorem is valid:

Theorem 5. Let the transformation T which consists of the successive applications of transformations T_1 and T_2 which are described by the equations (2.1) and (2.2) be such that the equation $x = T_2T_1x$ have at $\mu = 0$ an m-fold root. Then T has for $\mu = 0$ a complicated binomial cycle, which is divided for sufficiently small μ into not more than m binomial cycles of alternating stability. For even m the number of binomial cycles is even and the number of

stable cycles is equal to the number of unstable. For odd m the number of cycles is odd and is one larger than the number of cycles of stability of the generating cycle.

We now turn to the study of the bifurcation of complicated fixed points, the characteristic roots of which are equal to -1. In this case the transformation T for all sufficiently small μ has the fixed points M_1^*, M_2^*, in the vicinity of which the transformations T_1 and T_2 are described by the expressions (2.3) and (2.4). The transformation $T = T_2T_1$ in the vicinity of M_1^* has the form:

$$\bar{x} = \gamma_1 x + \ldots + \gamma_m x^m + \ldots, \tag{2.10}$$

where for $\mu = 0$

$$\gamma_1 = -1, \quad \gamma_2 = \ldots = \gamma_{m-1} = 0, \quad \gamma_m \neq 0, \quad m = \min(k_1, k_2),$$

while the transformation P^2 is

$$\bar{\bar{x}} = \nu_1 x + \nu_2 x^2 + \ldots + \nu_n x^n + \ldots, \tag{2.11}$$

where for $\mu = 0$

$$\nu_1 = 1, \quad \nu_2 = \nu_3 = \ldots = \nu_{n-1} = 0, \quad \nu_n \neq 0.$$

In (2.11) n = m, if m is odd, and n = m + 1 if m is even, since for $\mu = 0$

$$\bar{\bar{x}} = x - \gamma_m [1 + (-1)^{m-1}] x^m - \gamma_{m+1} [1 + (-1)^m] x^{m+1} + \ldots;$$

$$\nu_s = \sum_{l=1}^{s} \gamma_l \Big(\sum_{j_1 + \cdots + j_l = s} \gamma_{j_1} \cdots \gamma_{j_l} \Big) \quad (s = 1, 2, \ldots, m). \tag{2.12}$$

The coordinates of the fixed points of P^2 which differ from M_1^* are determined from the equations

$$(\nu_1 - 1) + \nu_2 x^* + \ldots + (\nu_n + \ldots)(x^*)^{n-1} = 0. \tag{2.13}$$

Equation (2.13) has for $\mu = 0$ a zero root of multiplicity (n-1). Consequently, (2.13) will have n - 1 roots for sufficiently small μ which approach zero as $\mu \to 0$ [15]. The Jacobian of the system (2.12) is equal to zero, since $\left.\frac{\partial \nu_s}{\partial \gamma_r}\right|_{\mu=0} = \begin{cases} 0 & , \quad r \neq s \\ -[1+(-1)^{s-1}] & , \quad r = s \end{cases}$

and the system (2.12) is not solvable relative to γ_i. In the proof of theorem 5, the essential point was that we could arbitrarily change the roots of Eq. (2.7), inasmuch as to each set of roots of the equation (2.7), there corresponded a definite surface in the space of the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m$. One can also change the roots of Eq. (2.13) arbitrarily, since the system (2.12) is actually solvable relative to γ_i (one can reduce it to the form for which the Jacobian for $\mu = 0$ differs from zero). One can do the latter by dividing all equations with even s by $(1 + \gamma_1)$, under the condition that $(1 + \gamma_1)$ has the order of smallness not greater than μ.

Let us consider the Jacobian of the new set of equations. The odd rows consist of zeros except that the main diagonal consists of -2. The even rows to the right of the main diagonal are zeros, to the left are finite quantities or zeros and on the main diagonal are -1. Actually,

$$W = \frac{\partial}{\partial \gamma_r}\left(\frac{\nu_{2q}}{1+\gamma_1}\right)_{\mu=0} = \lim_{\mu \to 0}\left[\left(\sum_{j_1+\cdots+j_r=2q} \gamma_{j_1}\cdots\gamma_{j_r}\right)(1+\gamma_1)^{-1}\right] + \tag{2.14}$$
$$+ \left.\frac{\gamma_1}{1+\gamma_1}\frac{\partial}{\partial \gamma_r}(\gamma_{2q})\right|_{\mu=0},$$

if $r \neq 1$. The second component for $r \neq 2q$ is equal to zero. If

$r \geqslant 2q$, then even the first component is equal to zero. If $r < 2q$, then under the summation symbol at least one of the j_i is not equal to 1 in each component. Since these γ_{j_i} have the order of smallness not less than μ, then the limit in (2.14) is the quantity which is finite or zero. If $r = 2q$, then

$$W = \lim_{\mu \to 0} \left[\left(\gamma_1^{2q} + \gamma_1 \right) (1 + \gamma_1)^{-1} \right] = \lim_{\mu \to 0} \left[\gamma_1 (1 + \gamma_1 + \dots + \gamma_1^{2q-2}) \right] = -1.$$

Now let $r = 1$. Then

$$\gamma_{2q}(1+\gamma_1)^{-1} = \gamma_1\gamma_{2q}(1+\gamma_1+\dots+\gamma_1^{2q-2}) + (1+\gamma_1)^{-1} \sum_{i=2}^{2q-1} \gamma_i \left(\sum_{j_1+\dots+j_i=2q} \gamma_{j_1} \cdots \gamma_{j_i} \right). \tag{2.15}$$

The derivative of the first component on the right side of (2.15) for $\mu = 0$ is equal to zero or a finite quantity. Upon differentiation of the second component, a quantity is obtained in the denominator of the order of magnitude of μ^2. Under the summation there is in each component not less than two coefficients γ with signs different from 1, i.e., having the order of magnitude μ. Consequently, the limit of the derivative of the second component on the right hand side of (2.15) for $\mu \to 0$ is a finite quantity or zero. Thus the Jacobian of the new system for $\mu = 0$ differs from zero and the system can be solved relative to $\gamma_1, \gamma_2, \dots, \gamma_m + \dots$ for sufficiently small μ:

$$\gamma_1 = -\sqrt{\nu_1}, \quad \gamma_i = \varphi_i(\sigma_2, \nu_3, \sigma_4, \nu_5, \dots, \nu_n + \dots) \quad (i = 2, 3, \dots, n-1);$$
$$\gamma_n + \dots = \varphi_n(\sigma_2, \nu_3, \dots, \nu_n + \dots), \tag{2.16}$$

where

$$\sigma_j = \nu_j \left(1 - \sqrt{\nu_1}\right)^{-1} \quad (j = 2, 4, \dots, n-1).$$

It is now necessary to carry out a subdivision of the vicinity of the point $N(\nu_1, \nu_2, \ldots, \nu_m + \ldots)_{\mu=0}$ for sufficiently small μ into the region of existence of identical collections of f.p. P^2. If x_1^* is the root of Eq. (2.13), then $x_2^* = Px_1^*$ is also a root of (2.13), since $P^2 x_2^* = x_2^*$. This is valid both for real and for complex x_1^*. It follows from (2.10) that

$$x_2^* + x_1^* = (1 + \gamma_1) x_1^* + \ldots + \gamma_m (x_1^*)^m + \ldots . \tag{2.17}$$

For sufficiently small x_1^*, Eq. (2.17) is a quantity of order μx_1^*, with accuracy within which $x_2^* = -x_1^*$. Neglecting quantities of order μx_1^* and above, we obtain from (2.13) that $\nu_{2q} = 0$, since n - 1 is an even number. Now let

$$\delta_i = \nu_{2i+1}(\nu_n + \ldots)^{-1} \quad (i = 1, 2, \ldots, k-1); \quad \delta_0 = (\nu_1 - 1)(\nu_n + \ldots)^{-1}; \tag{2.18}$$
$$(x^*)^2 = y^*; \quad m - 1 = 2k.$$

Then (2.13) takes the form:

$$\delta_0 + \delta_1 y^* + \ldots + \delta_{k-1} (y^*)^{k-1} + (y^*)^k = 0 . \tag{2.19}$$

If (2.19) has real roots ξ_e and complex roots $\eta_j \pm i \zeta_j$ (j = 1, ..., p; l = 1, 2, ..., k - 2p), then δ_q are the polynomials (2.9). Setting one of the ξ_e in (2.9) equal to zero, t of the ξ_e being positive and the remainder negative, we obtain the equation of the surface R_t which divides the region G_t and G_{t+1} in the vicinity of the point $\Delta(\delta_0, \delta_1, \ldots, \delta_{k-1})_{\mu=0}$ (G_t is the region of existence of t positive real roots of Eq. (2.19), i.e., the region of existence of 2t fixed points of the transformation P^2 or t binomial cycles P). Evidently, all regions G_t about the point Δ ($\mu = 0$), and for sufficiently small μ, the

initially complicated fixed point M_1^* is divided into a simple fixed point and t binomial cycles if for this μ the point $H(\delta_0, \ldots, \delta_{k-1})$ falls on the surface G_t. It should be noted that when the point H passes from G_t to G_{t+1} through the surface R_t in a change of μ, then in the appearance of a new binomial cycle of the transformation P, the simple f.p. changes its stability, giving up the stability which it possessed in G_t to the new cycle. Therefore, if the point H is in the region G_t with even t, then the simple f.p. P possesses the stability of the generating point; if t is odd, then the stability of the simple f.p. is contrary to the stability of the generating f.p. For sufficiently small μ the quantity γ_1 is close to -1, while ν_1 is close to $+1$, and consequently the transformations P and P^2 are represented by monotonic functions in a sufficiently small region about the point M_1^*. Therefore, for sufficiently small μ the stable and unstable cycles alternate, since the f.p. of P^2 of different stability alternate and $x_2^* = -x_1^*$ with accuracy up to quantities of higher order of smallness (theorem 1). From the above follows

Theorem 6. Let the transformation T, consisting of successive applications of the transformations T_1 and T_2, which are described by equations (2.1) and (2.2) be such that the equation $x = (T_2 T_1)^2 x$ has an n-fold root for $\mu = 0$. Then n is odd and T has a complex binomial cycle for $\mu = 0$ which decomposes for sufficiently small μ into not more than one binomial and n - 1 four-term cycles. The cycles of different stability alternate. If the number of four-term

cycles is even then the binomial cycle possesses the stability of the generating binomial cycle and the number of stable and unstable four-term cycles are even. If the number of the four-term cycles is odd then among them there is one more cycle of stability of the generating cycle while the stability of the binomial cycle is opposite to the stability of the generating cycle.

The results of theorems 5 and 6 can be applied to the transformation T consisting of the successive applications of the transformations $T_1, \ldots, T_p$, which depend upon the parameter μ. In this case the cycles will not be binomial and four-termed, but p-termed and 2p-termed, respectively. Proof is completely analogous.

Let the transformation T of a straight line into a straight line have an n-fold fixed point M_1^*. The points $M_{i+1}^* = T^i M_1^*$ are also n-fold f.p. of the transformation T. If T in the vicinity of M_i^* is denoted by T_i, then the problem of the investigation of bifurcations of n-fold f.p. of the transformation T in transition across the surface N_{+1}, N_{-1} reduces to the problem of the study of bifurcations of f.p. of the transformation consisting of successive applications of the transformations $T_1, \ldots, T_n$. In this case the following theorems are valid.

Theorem 7. Let the transformation T, which depends on the parameter μ, have an m-fold fixed point M^* at the point $\mu = 0$, such that the equation $M^* = T^m M^*$ has an n-fold root. Then, for sufficiently small μ, the point M^* divides into not more than n m-fold

fixed points of the transformation T of alternating stability. For even n, the number of fixed points is even and the number of stable points is equal to the number of unstable. For odd n, the number of fixed points is odd and the number of f.p. of stability of the generating fixed point is one larger than the number of f.p. of the opposite stability.

Theorem 8. Let the transformation T which depends on the parameter μ have an m-fold fixed point M^* at $\mu = 0$, such that the characteristic root is equal to -1, and the equation $M^* = T^{2m}M^*$ has an n-fold root. Then n is odd and for sufficiently small μ, the point M^* decomposes into not more than one m-fold f.p. and n - 1 2m-fold f.p. Fixed points of different stability alternate. The fixed points of the stability which M^* possesses are larger by one.

3. DISCONTINUOUS PIECEWISE LINEAR TRANSFORMATION OF A STRAIGHT LINE INTO A STRAIGHT LINE

Let the transformation T have the form:

$$\bar{x} = Tx = \begin{cases} T_1 x = a + \lambda_1 x \ , & x < 0 \\ T_2 x = -b + \lambda_2 x \ , & x > 0 \end{cases}, \qquad (3.1)$$

where $a > 0$, $b > 0$ and $x = 0$ is the point of disruption of the continuity of T.

The transformation T is characterized by three parameters-- λ_1, λ_2 and $\Delta_1 = ab^{-1}$.

Let us consider the following cases: 1) $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$; 2) $\lambda_1 < 0$, $0 < \lambda_2 < 1$; 3) $\lambda_1, \lambda_2 < 0$; 4) $\lambda_2 < 0$, $0 < \lambda_1 < 1$. In

each case we carry out investigation of the point transformation and construct in the space of the parameters the regions of existence of fixed points of various types.

1) In the first case, the transformation T is a compressing one. It is not difficult to see that each successive iteration leads into the interval (-b, a), from which one can not leave. All are cycles of the transformation T by virtue of the fact that it is compressing, and stable.

Transformations of the form $T_2^m T_1^n$, where m and n are larger than 1, are impossible, since in the opposite case it would be $T(-b) < 0$, $Ta > 0$ simultaneously, whence $\lambda_1 \lambda_2 > 1$ which contradicts the assumption $\lambda_1 \lambda_2 < 1$.

We divide the cycles of the transformation T into classes of complexity. We call cycles of the first complexity those which contain f.p. of transformation of the type $T_2^n T_1$, $T_1^n T_2$.

Transformations of the type $T_2^n T_1$ are possible if $T(-b) > 0$, i.e., $\Delta_1 > \lambda_1$. In this case we consider the transformation P_1 of the interval (-b, 0) into itself, so that each point of this interval, as the result of application of P_1, turns again into (-b, 0). Here there can be two cases. In the first case P_1, which coincides with $T_2^n T_1$, is continuous in (-b, 0), transforms (-b, 0) into itself and, by virtue of the fact that it is compressing, has a single fixed point for each n. In the second case $T_2^n T_1$ transforms (-b, 0) into an interval containing the point $x = 0$, and for part of the form, where $x > 0$,

it is necessary to apply T_2 again in order to transform this part into $(-b, 0)$. Inasmuch as T_1 and T_2 are compressing, $(-b, 0)$ will be transformed into itself. In this case P_1 is a discontinuous transformation, which decomposes into $T_2^n T_1$ and $T_2^{n+1} T_1$ (see Fig. 1). The transformation $T_2^n T_1$

$$\overline{x}_{2n} = \lambda_2^n a - b(1 - \lambda_2^n)(1 - \lambda_2)^{-1} + \lambda_1 \lambda_2^n x \qquad (3.2)$$

has the fixed point

$$x_{2n}^* = [\lambda_2^n a - b(1 - \lambda_2^n)(1 - \lambda_2)^{-1}](1 - \lambda_1 \lambda_2)^{-1}. \qquad (3.3)$$

The division of the region of the values of the parameter $\Delta_1 = ab^{-1}$ into the region of the existence of the fixed points x_{2n}^* of the transformation $T_2^n T_1$ is determined by the condition $-b < x_{2n}^* < 0$:

$$(1 - \lambda_2^{n-1})(1 - \lambda_2)^{-1} \lambda_2^{1-n} + \lambda_1 < \Delta_1 < (1 - \lambda_2^n) \lambda_2^{-n} (1 - \lambda_2)^{-1}. \quad (3.4)$$

This partition is an infinite sequence of intervals of length $(1 - \lambda_1 \lambda_2) \times \lambda_2^{-n}$ which go off to infinity as $n \to \infty$. The distance between intervals is λ_1. Transformations of the form $T_1^n T_2$ are possible if $Ta < 0$, i.e., $\Delta_1 < \lambda_1^{-1}$. In this case one can use the previous results, substituting x_{2n}^* for $-x_{1n}^*$, where x_{1n}^* is the coordinate of the f.p. of the transformation $T_1^n T_2$, and exchanging λ_1 and λ_2, a and b. Actually, if in (3.1) we make the substitution $x = -y$, $\overline{x} = -\overline{y}$, then the resultant transformation will differ from (3.1) only by the places of a, b, λ_1, λ_2. Thus the region of the existence of f.p.

$$x_{1n}^* = [a(1 - \lambda_1^n)(1 - \lambda_1)^{-1} - b\lambda_1^n](1 - \lambda_1 \lambda_2)^{-1} \qquad (3.5)$$

is

$$(1 - \lambda_1^{n-1}) \lambda_1^{1-n} (1 - \lambda_1)^{-1} + \lambda_2 < \Delta_1^{-1} < (1 - \lambda_1^n) \lambda_1^{-n} (1 - \lambda_1)^{-1}. \quad (3.6)$$

Here the region of values of the parameter Δ_1 is decomposed into an infinite sequence of nonintersecting intervals, which converge for $n \to \infty$ to the point $\Delta_1 = 0$. After division of the regions of existence of cycles of first complexity on the axis Δ_1, there remain "empty" intervals, in each of which either the discontinuous transformations P_1 or P_2 which decompose into the transformations $T_2^n T_1$, $T_2^{n+1} T_1$ or $T_1^{n+1} T_2$, $T_1^n T_2$ act, respectively. The first takes place in intervals bounded with the intervals of existence of the f.p. of $T_2^n T_1$, the second in intervals bounded with intervals of existence of f.p. of $T_1^n T_2$.

We call cycles of second complexity cycles of first complexity relative to transformations taking place in each "empty" interval. They contain f.p. of the transformations $(T_2^{n+1} T_1)^m (T_2^n T_1)$, $(T_2^n T_1)^m (T_2^{n+1} T_1)$, $(T_1^n T_2)^m (T_1^{n+1} T_2)$, $(T_1^{n+1} T_2)^m (T_1^n T_2)$. We consider the transformation P_1 (Fig. 1) which, after transfer of the origin of the coordinates to the point O_1, has the form:

$$\bar{y} = \Pi_1 y = \begin{cases} a_1 + \lambda_{12} y, & y < 0 \\ -b_1 + \lambda_{22} y, & y > 0 \end{cases}, \tag{3.7}$$

where

$$a_1 = \left[\lambda_2^n a - b\left(1 - \lambda_2^n\right)(1 - \lambda_2)^{-1}\right] \lambda_1^{-1} \lambda_2^{-n};$$

$$b_1 = b + \left[b\left(1 - \lambda_2^n\right)(1 - \lambda_2)^{-1} - \lambda_2^{-n} a\right] \lambda_1^{-1} \lambda_2^{-n};$$

$$\lambda_{12} = \lambda_1 \lambda_2^n, \; \lambda_{22} = \lambda_1 \lambda_2^{n+1}, \; \lambda_{12} < 1, \; \lambda_{22} < 1. \tag{3.8}$$

The region of values $\Delta_2 = a_1 b_1^{-1}$ is divided into the region of existence of f.p. of the transformations $(T_2^{m+1} T_1)^m (T_2^m T_1)$, $(T_2^m T_1)^m (T_2^{m+1} T_1)$ the same as the region of values Δ_1 into the region of existence of the f.p. of $T_2^n T_1$ and $T_1^n T_2$. Thus, on the ray Δ_2, there is a sequence of non-overlapping intervals, infinite on both sides, on the left converging to $\Delta_2 = 0$, and on the right diverging to infinity. From (3.8)

$$\Delta_1 = (1 - \lambda_2^n)(1 - \lambda_2)^{-1} \lambda_2^n + \lambda_1 \Delta_2 (1 + \Delta_2)^{-1} \tag{3.9}$$

and in each "empty" interval on the ray Δ_1, in which P_1 occurs (and P_1 occurs in "empty" intervals for which $\Delta_1 > 1$), there is a set of non-overlapping intervals, infinite on both sides, converging to both ends of the corresponding "empty" intervals--regions of existence of cycles of second complexity.

Intervals in which the transformation P_2 occurs are divided in similar fashion. In this case, Δ_1 and Δ_2 are connected by the following relation:

$$\Delta_1^{-1} = (1 - \lambda_1^n)(1 - \lambda_1)^{-1} \lambda_1^{-n} + \lambda_2 (1 + \Delta_2)^{-1}. \tag{3.10}$$

Cycles of third complexity, fourth complexity, ..., Nth complexity are determined analogously. Let the division of the axis into a region of existence of cycles up to N - 1 complexity, inclusively, be carried out. After this, there remains a denumerable sequence of "empty" intervals on Δ_1, each of which has a boundary with intervals of existence of cycles of complexity, 1, 2, ..., N - 1. Let us consider one of the "empty" intervals, which we shall denote by

Δ_{1i}. The region to the left of the interval is a region of existence of f.p. of some transformation T_i, on the right, of a transformation T_{i+1}. In Δ_{1i} there is a discontinuous transformation P_i, which decomposes into T_i and T_{i+1}. Cycles of complexity N we call cycles of first complexity of transformation P_i. One of the elements of the cycle of complexity N is the f.p. of the transformation $T^n_{i+1} T_i$ or $T^n_i T_{i+1}$. The region of existence of these f.p. forms for each i a sequence of non-overlapping intervals on the axis Δ_N that is infinite on both sides. On the axis Δ_1, to each such sequence there corresponds a sequence of non-overlapping intervals, infinite on both sides, inclosed in the corresponding interval Δ_{1i} and converging on the left and right to its ends. In this case the translation of the subdivision of the Δ_{k+1} axis to the Δ_k axis is carried out according to the formulas

$$\Delta_k = \left(1 - \lambda_{2k}^{p_k}\right)(1 - \lambda_{2k})^{-1} \lambda_{2k}^{-p_k} + \lambda_{1k} \Delta_{k-1} (1 + \Delta_{k+1})^{-1} \tag{3.11}$$

or

$$\Delta_k^{-1} = \left(1 - \lambda_{1k}^{p_k}\right)(1 - \lambda_{1k})^{-1} \lambda_{1k}^{-p_k} + \lambda_{2k} (1 + \Delta_{k+1})^{-1}. \tag{3.12}$$

If the translation is carried out in the interval on Δ_k for which $\Delta_k > 1$, then (3.11) applies, if on an interval for which $\Delta_k < 1$, then we use (3.12).

Thus the space of the parameters λ_2, Δ_1, λ_1 is divided into a region of existence of cycles of various complexity (see Fig. 2). Cycles of the first complexity correspond to a denumerable sequence of non-overlapping regions, infinite on both sides, converging on the

left to $\Delta_1 = 0$, on the right, diverging to infinity. In each "empty" region a similar sequence of regions of existence of cycles of second complexity is inclosed, infinite on both sides, converging to the boundary surfaces of the corresponding "empty" region. In each new "empty" region there are inclosed similar sequences of regions of existence of cycles of third complexity, and so forth.

Let us consider the set M of closed intervals on the lines λ_2 = const, λ_1 = const, $\Delta_1 > 0$; these intervals are regions of existence of cycles of all possible complexities. The set M is denumerable since it consists of a denumerable set of pairwise non-intersecting denumerable sets. If the set M is removed from the set of values of the parameter Δ_1, then, as follows from (3.11) and (3.12), the difference is a non-empty, unbounded, everywhere dense, complete set R. Therefore the set R is similar to the Kantor discontinuum, and each point of this set is a limit for the end points of the intervals forming the set M.

The motion of the point M on the plane $x, \bar{x}$ in the process of iteration is stable according to Poisson if $\Delta_1 \in$ R. Actually, closed trajectories do not exist, since a cycle of some finite complexity would have existed in the contrary case. The point M in the motion, as was shown above, does not depart from the interval (-b, a). Consequently, there exists at least one point of compression in an arbitrarily small region about which the point M falls an unbounded number of times. This motion is not periodic and takes place in the

following fashion. Let the motion begin from the point x_0. The following points of the sequence of iteration are obtained by successive application of some transformation G--the product of a finite number of transformations T_1 and T_2: Gx_0, G^2x_0 and so on up to some point $G^n x_0 = y$. Then the points are obtained by successive application of the transformation G_1--the product of a larger number of transformations T_1 and T_2 then in G: G_{1y}, $G^2{}_{1y}, \ldots$ up to the point $G_1^m y = z$ and so on. In this case the number of transformations T_1 and T_2 entering into a definite G_i is greater than into G_{i-1}, and the transformation G_i is replaced through a finite number of iterations by a more complex transformation G_{i+1} where i increases without limit.

2) In the second case there can be only fixed points of the transformation $T_2^n T_1$. They are stable if $|\lambda_1|\lambda_2^n < 1$ [see (3.2)]. The point M in the plane $x, \bar{x}$, by means of a finite number of iterations, transforms to the interval $[-b, T_1(-b)]$, from which it cannot depart.

The fixed point (3.3) of the transformation (3.2) has its own region of existence (3.4). These regions for any pair λ_1, λ_2 form an infinite sequence of intervals of length $\lambda_2^{-n} - \lambda_1$, each of which overlaps its neighbor by an amount $|\lambda_1|$(Fig. 3).

If λ_1 is such that

$$\sum_{k=1}^{m-1} \lambda_2^{-n-k} < |\lambda_1| < \sum_{k=1}^{m} \lambda_2^{-n-k}, \tag{3.13}$$

then for $\Delta_1 \in (\lambda_2^{-n})$ there exists $x^*_n, x^*_{n+1}, \ldots, x^*_{n+m-1}$, if $\Delta_1 \in (n+m)$; moreover, there also exists a point x^*_{n+1} if $\Delta_1 \notin (n+m)$. Here $(n+m)$ is the interval of the number $n+m$, (λ_2^{-n}) is the interval between the right ends (n) and $(n-1)$ (see Fig. 3), while x^*_{n+k} is the fixed point of the transformation $T_2^{n+k}\,T_1$. The inequalities (3.13) show that (λ_2^{-n}) belongs to each interval $(n+k)$ $(k = 1,2,\ldots,m-1)$, that $(n+m)$ covers (λ_2^{-n}) either ~~as~~ in whole or in part, and that (n) and $(n+m+1)$ have no common points. By virtue of the left side of (3.13), $|\lambda_1| > \lambda_2^{-n-k}$ $(k = 1,\ldots,m-1)$, and consequently $|\lambda_1| > \lambda_2^{-n}$, that is, $x^*_{n+k}(k = 0,\ldots,m-1)$, are unstable. One can say the following about the stability of x^*_{n+m}

$$f_1 = \lambda_2^{-n-m}; \quad f_2 = \sum_{k=1}^{m-1} \lambda_2^{-n-k}; \quad f_3 = \sum_{k=1}^{m} \lambda_2^{-n-k};$$

$$f_1 - f_3 = \lambda_2^{-n-m}\left(1 - 2\lambda_2 + \lambda_2^m\right)(1-\lambda_2)^{-1},$$

and let $\lambda_2(m)$ be the root of the equation $f_1 = f_3$, which belongs to the interval $(0,1)$. It is evident that $\lambda_2(m) > 0.5$. $f_1 > f_3$ if $m = 2$, and also if $m \geq 3$ and $0 < \lambda_2 < \lambda_2(m)$. $f_1 < f_3$ if $m \geq 3$ and $\lambda_2(m) < \lambda_2 < 1$. In the first case, the point x^*_{n+m} appears as a stable point in the series x^*_{n+k} $(k = 0, 1, \ldots, m-1)$ for increase in $|\lambda_1|$, since $(n+m)$ overlap with (n) sooner than $|\lambda_1|$ becomes larger than λ_2^{-n-m}. In the second case, x^*_{n+1} appears in the series x^*_{n+k} as an unstable point, since at the moment of contact of $(n+m)$ with (n), $|\lambda_1| > \lambda_2^{-n-m}$. $f_1 < f_3$ for any $m > 0$. Consequently, upon increase

of $|\lambda_1|$, the point x^*_{n+m} becomes unstable up to the appearance of the intersection of the cuts (n) with (n + m + 1), i.e., up to the appearance of the possibility of the existence of fixed points x^*_{n+k} ($k = 0, 1, \ldots, m + 1$) simultaneously.

As a result, the following takes place. Let $0 < \lambda_2 < 0.5$. If $\Delta_1 \in (\lambda_2^{-n})$, then for $|\lambda_1| < \lambda_2^{-n}$, where λ_1 satisfies the condition

$$|\lambda_1| < (1 - \lambda_2^n)\lambda_2^{-n}(1 - \lambda_2)^{-1} - \Delta_1 \tag{3.14}$$

for fixed Δ_1 is such that $\Delta_1 \bar{\in}$ (n + 1), there corresponds the single stable fixed point x^*_n. When (3.14) becomes an equality with increase in $|\lambda_1|$, another fixed point x^*_{n+1} appears in addition to x^*_n, the stability of which follows from the inequality $|\lambda_1| < \lambda_2^{-n} < \lambda_2^{-n-1}$. Then for $|\lambda_1| = \lambda_2^{-n}$ a change in the stability of x^*_n takes place and for $|\lambda_1| = \lambda_2^{-n-1}$ a change in the stability of x^*_{n+1}. When $|\lambda_1| = \lambda_2^{-n-1} + \Delta_1$, the stable point x^*_{n+2} appears which changes stability for $|\lambda_1| = \lambda_2^{-n-2}$, ... For $|\lambda_1| = \lambda_2^{-n-k} +$ the stable fixed point x^*_{n+k+1} appears, which is the only fixed point in the whole series of fixed points x^*_{n+p} ($p = 0, 1, \ldots, k + 1$) which exists simultaneously for the given . For $|\lambda_1| = \lambda_2^{-n-k-1}$ the point x^*_{n+k+1} also becomes unstable, etc.

Now let $0.5 < \lambda_2 < 1$. Then $\lambda_2(m)$ form an infinite sequence converging to $\lambda_2 = 0.5$ for $m \to \infty$. With increase of $|\lambda_1|$ m increases but $\lambda_2(m)$ decreases. Therefore, whatever $\lambda_2 \in (0.5; 1)$, m certainly reaches such values that $\lambda_2(m)$ becomes less than λ_2

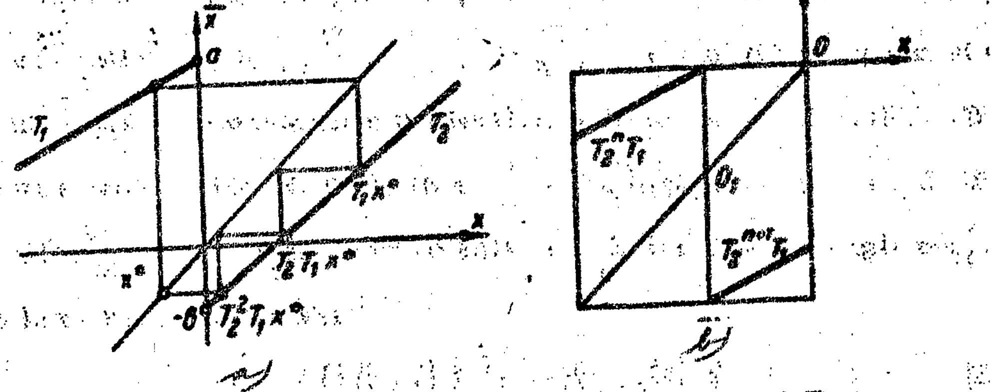

Fig. 1. The transformation schemes T and P_1

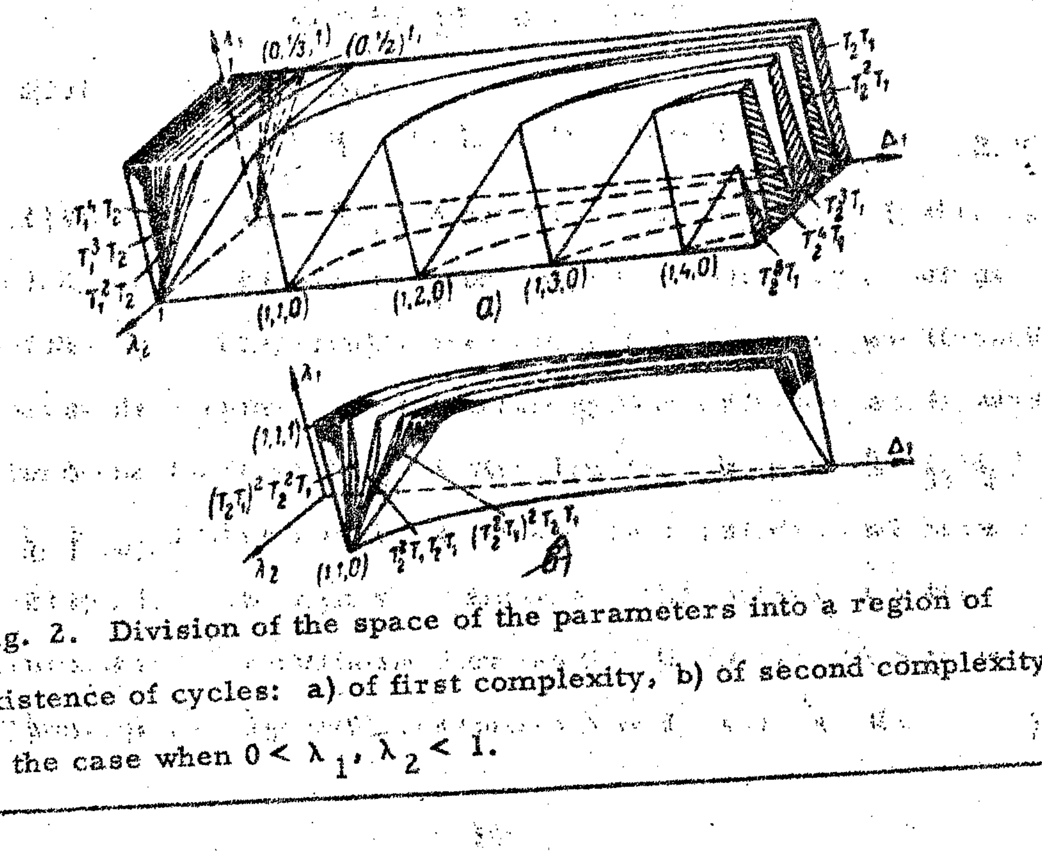

Fig. 2. Division of the space of the parameters into a region of existence of cycles: a) of first complexity, b) of second complexity in the case when $0 < \lambda_1, \lambda_2 < 1$.

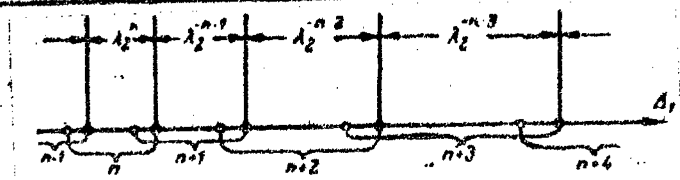

Fig. 3. Location of the regions of existence of various cycles on the

the axis.

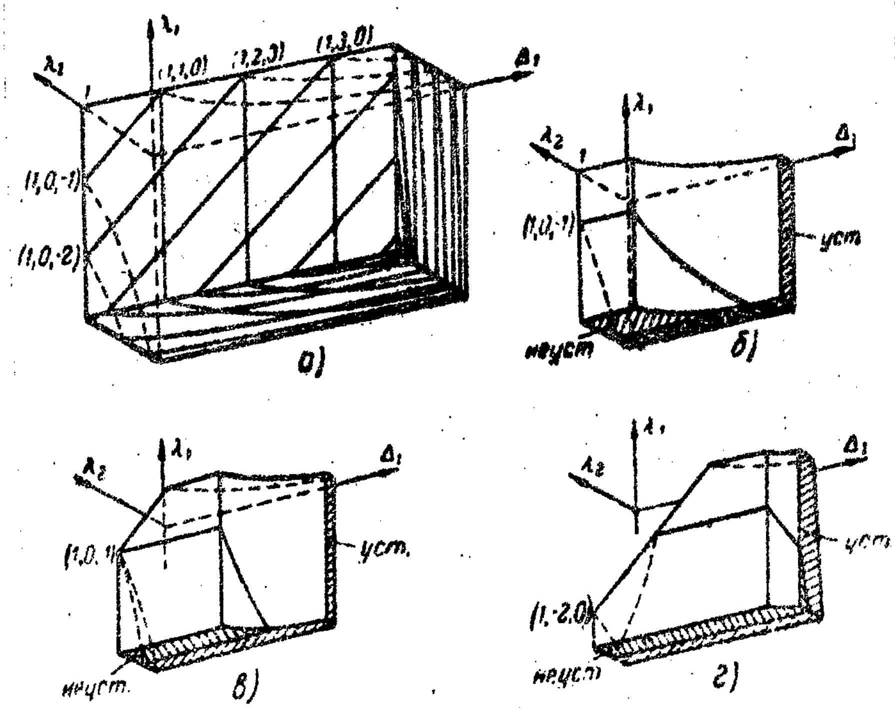

Fig. 4. Division of the space of the parameters into the region of existence of various cycles (a) and the region of existence of fixed points of the transformations b) T_2T_1, c) $T_2^2T_1$, d) $T_2^3T_1$ in the case when $\lambda_2 < 0$, $0 < \lambda_2 < 1$.

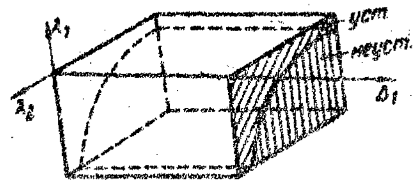

Fig. 5. Division of the space of the parameters in the case when $\lambda_1, \lambda_2 < 0$.

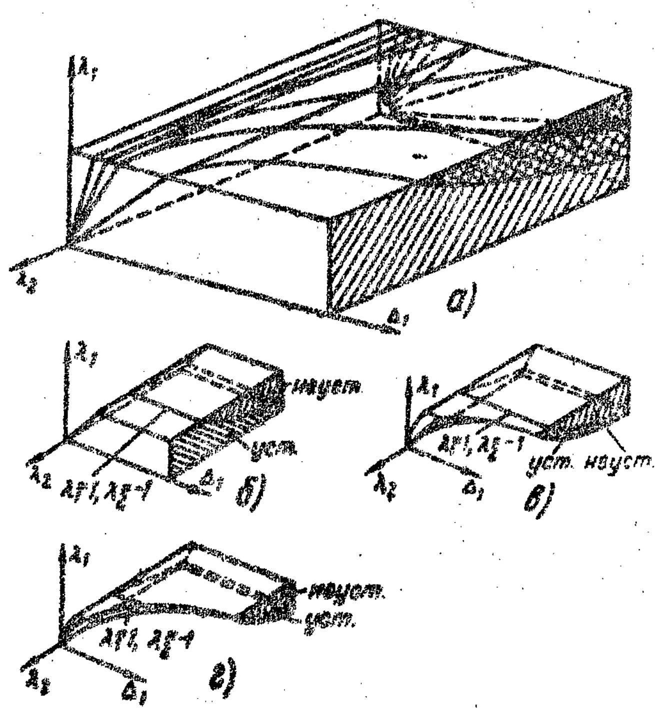

Fig. 6. Division of the space of the parameters into the region of existence of various cycles (a) and the region of existence of the fixed points of the transformations b) T_1T_2, c) $T_1^2T_2$, d) $T_1^3T_2$ in the case when $\lambda_2 < 0$, $0 < \lambda_1 < 1$.

upon increase of $|\lambda_1|$. While $\lambda_2(m) > \lambda_2$, the bifurcations of the fixed points for increase of $|\lambda_1|$ takes place as described above. For λ_{10} let $\lambda_2(m)$ become more than λ_2. Then, for arbitrary $\lambda_1 < \lambda_{10}$, there will exist only unstable fixed points. Thus, if $\lambda_2^{-n-k} + \Delta_1 < |\lambda_1| < \lambda_2^{-n-k-1}$, when $\lambda_2 < \lambda_2(m)$, among the whole series of points x^*_{n+p} $(p = 0, 1, \ldots, k+1)$, which exist simultaneously, only x^*_{n+k+1} is stable (with the exception of the case $k = 1$, when there can exist two stable f.p., x^*_n and x^*_{n+1}), i.e., the cycle $y_1, y_2, \ldots, y_{s+2}, \ldots, y_{n+k}$ is stable, where $y_1 = x^*_{n+k+1}$ and $y_{s+2} = T_2^s T_1 y_1$. If, for $\lambda_2 < \lambda_2(m)$, $\lambda_2^{-n-k} < |\lambda_1| < \lambda_2^{-n-k} +$ and for all λ_1 in the case $\lambda_2 < \lambda_2(m)$ there are no stable fixed points for a given Δ_1. However, in this case there is a bounded region $[-b, T_1(-b)]$, in which all the sequences of the iterations enter the point M, and the point N cannot depart from them. The motions here are non-periodic and stable according to Poisson. Division of the space of the parameters λ_2 Δ, λ_1 into the region of existence of station of fixed points of the transformation T is shown in Fig. 4.

3) In the third case the transformation, T_2T_1

$$x_2 = \lambda_1\lambda_2 x - b + \lambda_2 a$$

maps the semiaxis $x < 0$ into itself. If $\lambda_1\lambda_2 < 1$, there exists a single f.p. $x^* = (\lambda_2 a - b)(1 - \lambda_1\lambda_2)^{-1}$. In the case $\lambda_1\lambda_2 > 1$, all the sequences of the iteration diverge. The division of the space of the parameters is shown in Fig. 5.

4) The results of case 4 coincide with the results of case 2 if in the latter we transpose a and b, λ_1 and λ_2, as shown in case 1. The division of the space of the parameters is pictured in Fig. 6.

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THE STABILITY IN THE LARGE OF THE EQUILIBRIUM STATE OF RELAY SYSTEMS

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(Abstract) For one form of relay systems of automatic regulation an effective method is proposed for the setting up of criteria of stability for any initial excitation, based on the strict application of the method of Lyapunov functions for these systems.

1. A very widespread class of relay systems of regulation is considered, whose excited motion is described by the equations (relative representations):

$$
\begin{aligned}
&Q(p)\Sigma(p) = R(p)\,Y(p); \qquad (1)\\
&Y(p) = -L\{k_p f(\sigma)\};\\
&f(\sigma) = \operatorname{sgn}\sigma; \quad k_p > 0,
\end{aligned}
$$

for the following assumptions:

a) the transmission function of the linear part of the system has the form

$$W(p) = \frac{R(p)}{Q(p)} = \frac{b_n p^{n-1} + \ldots + b_2 p + b_1}{a_{n+1} p^n + \ldots + a_2 p + a_1}, \quad (2)$$

where the coefficients $a_1, \ldots, a_{n+1}$, $b_1, \ldots, b_n$ are constants,

$$b_1 > 0, \quad a_{n+1} > 0; \quad (3)$$

b) the real parts of all roots of the equation

$$a_{n+1} \lambda^n + \ldots + a_2 \lambda + a_1 = 0 \quad (4)$$

are negative, so that $a_1, \ldots, a_n > 0$.

Moreover, we limit ourselves to the case in which

$$b_n = 0, \quad b_{n-1} > 0. \quad (5)$$

For the system (1), the stability criteria in the small were found in [1] by a non-rigorous, and in [2,3] by a rigorous method. Sufficient conditions for stability in the large can be sought for (1), for example, by following Lur'e [4,5], if the applicability of the second method of Lyapunov to systems (1) with a discontinuous right hand side is first shown.

In the present note it is assumed (with rigorous justification) that there is an effective method of investigation of relay systems (1) for the stability in the large, similar in essentials to a modification of the method of Malkin ([5], pp. 163-164) which was applied in [6] for the study of absolute stability. The case of (5) is considered below, to which the results of [6], where the assumptions $a_1 = 0$, $b_n > 0$ were essentially employed, are not directly applied. Yielding somewhat in generality, we turn aside

From the invariant form of the exposition given in [6], and take the equations of the relay systems under study in the form (1), which is most frequently used in the theory of regulation. In this case the criteria for stability will be expressed directly in terms of the coefficient of the transfer function (2). The character of the roots of equation (4) is unessential in the framework of limitation c). The Lyapunov functions are constructed by a method which has an explicit geometric interpretation and which does not require a reduction of the initial equations of motion to conconical form [4,5].

2. Denoting by X and F(x) the columns, and by X' and F(x)' the corresponding rows of an n-dimensional vector, we write (1) in the form

$$dX/dt = F(x);$$

$$F(x) = \begin{cases} F_{+}(x) \ \text{при} \ \sigma = \sum_{k=1}^{n-1} b_k x_k \geq 0 \\ F_{-}(x) \ \text{при} \ \sigma = \sum_{k=1}^{n-1} b_k x_k < 0 \end{cases}, \tag{6}$$

where

$$X = (x_1, \ldots, x_n)';$$

$$F_{+}(x) = \left(x_2, \ldots, x_n, -\sum_{r=1}^{n} \frac{a_r}{a_{n-1}} x_r - \frac{k_p}{a_{n-1}} \right)'; \tag{7}$$

$$F_{-}(x) = \left(x_2, \ldots, x_n, -\sum_{r=1}^{n} \frac{a_r}{a_{n-1}} x_r + \frac{k_p}{a_{n-1}} \right)';$$

Below, together with the equations

$$dX/dt = F_{+}(x); \tag{8}$$

$$dX/dt = F_{+}(x), \tag{9}$$

which are assumed to be given throughout all phase space x,

there is considered the stable linear system

$$dX/dt = \left(x_2, \ldots, x_n, -\sum_{r=1}^{n} a_r a_n{}^1{}_1 x_r\right)', \qquad (10)$$

obtained from (6)-(7) for $k_p \equiv 0$, and also the functions σ_r $(x_1, \ldots x_n)$ - and $\dot{\sigma}_+$ $(x_1, \ldots, x_n)$ (or $\dot{\sigma}_-$, $\ddot{\sigma}_-$)which are total derivatives in time of the corresponding functions $\sigma_+ (x_1, \ldots, x_n)$ and $\dot{\sigma}_+ (x_1, \ldots x_n)$ (or σ_-, $\dot{\sigma}_-$) in view of the system (8) (or the system (9)). Direct calculation gives:

$$\dot{\sigma}_+ (x_1, \ldots, x_n) = \sum_{k=1}^{n-1} b_k x_{k+1} = \dot{\sigma}_- (x_1, \ldots, x_n) = \dot{\sigma}; \qquad (11)$$

$$\ddot{\sigma}_+ = \sum_{k=1}^{n} (b_{k-2} - b_n + a_n{}^1{}_1 a_k) x_k - b_{n-1} a_n{}^1{}_1 k_p; \qquad (12)$$

$$\ddot{\sigma}_- = \sum_{k=1}^{n} (b_{k-2} - b_{n-1} a_{n+1}^{-1} a_k) x_k + b_{n-1} a_{n+1}^{-1} k_p$$

so that $(b_{-1} = b_0 = 0)$,

$$\ddot{\sigma}_+ (x_1, \ldots, x_n) > \ddot{\sigma}_- (x_1, \ldots, x_n). \qquad (13)$$

Moreover, the system

$$dX/dt = F_0 (x) = \left(x_2, \ldots, x_n, -b_n{}^1{}_1 \sum_{k=1}^{n} b_{k-2} x_k\right)', \qquad (14)$$

is used below, and is equivalent to the equation

$$\dot{\sigma} = b_{n-1} d^{n-2} x_1/dt^{n-2} + \ldots + b_2 dx_1/dt + b_1 x_1 = 0.$$

The trajectories of this system are distributed over the set

$$\sigma(x_1, \ldots, x_n) \quad c(x_1, \ldots, x_n) = 0.$$

Finally, we agree to denote by $X(P,t)$, $X_+(P,t)$, $X_-(P,t)$, $X_0(P,t)$ the trajectories of the corresponding sets (6)-(7), (8), (9), (12) which, at $t = 0$, pass through the point P of phase space. By the vicinity of the phase point we shall understand its vicinity in the space x.

3. Following [3,7-9], we shall give a rigorous definition of the trajectory $X(P,t)$, which is suitable for the whole space x and which satisfies physical considerations relative to the dynamics of the systems (6)-(7) upon fulfillment of the limitations (3), (5).

In the subspace $\sigma > 0$ we shall assume $X(P,t) = X_+(P,t)$; similarly, in the subspace $\sigma < 0$, $X(P,t) = X_-(P,t)$.

In accordance with (11), the trajectories $X_+(P_s,t)$ and $X_-(P_s,t)$ at any point P_s of the set $\sigma = 0$, $\dot\sigma \neq 0$ intersect the reversing surface in the same direction. We shall determine the trajectory $X(P_s,t)$ by joining together in continuous fashion at the point P_s the half-trajectories $X_+(P_s,t)$ and $X_-(P_s, t)$, which lie (even close to P_s) in the respective regions $\sigma > 0$ and $\sigma < 0$.

The trajectory $X_+(P_+,t)$ (or $X_-(P_-,t)$), where $P_+(P_-)$ is a point of the set $\sigma = \dot\sigma = 0$, $\dot\sigma_- > \dot\sigma_+ > 0$ ($\sigma = \dot\sigma = 0$, $0 > \dot\sigma_- < \dot\sigma_+$) is located in the subspace $\sigma > 0$ ($\sigma < 0$), by virtue of the relation $\ddot\sigma_+(P_+) > 0$, ($\ddot\sigma_-(P_-) < 0$), at least in a certain region about the point $P_+(P_-)$. In this region, $X(P_+,t)$ (or $X(P_-,t)$) is

considered as coinciding with $X_+(P_+,t)$ $(X_-(P_-,t))$. Assuming that the neighborhood is sufficiently small, we see that in it $X_-(P_+,t)$ (or $X_+(P_-,t)$) lies in the region $\sigma > 0$ $(\sigma < 0)$ as a consequence of the inequality $\ddot{\sigma}_-(P_+) > 0$ $(\ddot{\sigma}_+ (P_-) < 0)$, so that the oneness of $X(P,t)$ at the point P_+ (P_-) is not violated.

In a sufficiently small vicinity of any point Q of the set

$$\sigma = \dot{\sigma} = 0, \qquad \ddot{\sigma}_- > 0 > \ddot{\sigma}_+ \tag{15}$$

the trajectory $X_+(Q,t)$ $(X_-(Q,t))$ is found in the halfspace $\sigma < 0$ $(\sigma > 0)$. In this region we assume [3] that $X(Q,t) = X_0(Q,t)$. Such a definition of the vector phase velocity $F(Q) = F_0(Q)$ has a clear geometric interpretation: if we assume the vectors $F_+(Q)$, $F_-(Q)$, $F_0(Q)$ applied at the point $Q(x^*_1,\dots,x^*_n)$, then the end of the vector $F_0(Q) = F(Q)$ is determined by the point of intersection of the plane $\dot{\sigma} = 0$ and the line joining the ends of the vectors $F_+(Q)$ and $F_-(Q)$. This interpretation makes the obvious relation

$$f_n^-(Q) \quad f_n^0(Q) > f_n^+(Q) \tag{16}$$

between the projections

$$
\begin{aligned}
f_n^-(Q) &= -\sum_{k=1}^{n} a_k a_{n+1}^{-1} x_k + k_p a_{n+1}^{-1}; \\
f_n^0(Q) &= -\sum_{k=1}^{n-2} b_k b_{n-1}^{-1} x_{k+2}; \\
f_n^+(Q) &= -\sum_{k=1}^{n} a_k a_{n+1}^{-1} x_k - k_p a_{n+1}^{-1}
\end{aligned}
\tag{17}
$$

of the vectors $F_-(Q)$, $F_0(Q)$, $F_+(Q)$ on the OX_n axis, inasmuch as the projections of these vectors on each of the axes OX_k $(k = 1,\dots,n-1)$ coincide (see (7) and (14)). The unity of $X(P,t)$ at the points Q is preserved in the same way as at the points P_s.

Finally, let us consider the set $\sigma = \dot{\sigma} = 0, \ddot{\sigma}_- > 0 = \ddot{\sigma}_+$ (the case $\sigma = \dot{\sigma} = 0,\ 0 = \ddot{\sigma}_- > \ddot{\sigma}_+$ is treated similarly). In accord with (7), (14), (12), at an arbitrary point R of this set,

$$F_+(R) = F_0(R), \tag{18}$$

so that

$$\frac{d^m}{dt^m}\left[\sigma_+(R)\right]_+ = \frac{d^m}{dt^m}\left[\sigma_+(R)\right]_0, \tag{19}$$

where $\frac{d^m}{dt^m}\left[\sigma_+(R)\right]_+$ and $\frac{d^m}{dt^m}\left[\sigma_+(R)\right]_0$ are the total time derivatives of order m of the functions $\sigma_+ \ (x_1,\dots,x_n)$, computed at the point R in correspondence with the systems (8) and (14). It is seen from (19) that the positive halftrajectories of $X_+(R,t)$ and $X_0(R,t)$ in a sufficiently small vicinity R, lie simultaneously either in the region $\sigma_+ \geq 0$ or in the region $\sigma_+ < 0$. In this region, we have in the first case $X(R,t) = X_+(R,t)$, and in the second, $X(R,t) = X_0(R,t)$. Inasmuch as the identical situation takes place for the negative halftrajectories $X_+(R,t)$ and $X_0(R,t)$, the unity of the solution of $X(R,t)$ at the points R is not destroyed.

4. The continuation of the solution of $X(P,t)$ determined

above is obvious fot $t \to \infty$; the continuity according to the original data is easily observed with account of (18). Therefore, the reflection of $X(P,t)$ satisfies all the axioms [10] of a dynamic system. The trajectory of $X(P,t)$ is nondifferentiable only at points of entry onto the reversing surface, i.e., at a denumerable number of moments of time. Many of the results of the ordinary theory of stability are immediately extended (with insignificant changes) to such dynamical systems. One such theorem is that of Barbashin and Krasovskiĭ [11] on the stability as a whole (see below).

5. We consider the functions

$$V_1(x_1, \dots, x_n) = \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j \qquad (\alpha_{ij} = \alpha_{ji}); \tag{20}$$

$$V'(x_1, \dots, x_n) = V_1(x_+, x_2, \dots, x_n) = V_1(x_1, \dots, x_n) +$$

$$+ 2k_p a_1^{-1} \sum_{k=1}^{n} \alpha_{1k} x_k + k_p^2 a_1^{-2} \alpha_{11} \qquad (x_+ = x_1 + k_p a_1^{-1}); \tag{21}$$

$$V''(x_1, x_2, \dots, x_n) = V_1(x_-, x_2, \dots, x_n) = V_1(x_1, \dots, x_n) -$$

$$- 2k_p a_1^{-1} \sum_{k=1}^{n} \alpha_{1k} x_k + k_p^2 a_1^{-2} \alpha_{11} \qquad (x_- = x_1 - k_p a_1^{-1})$$

and their total derivatives $\dot V_1$, $\dot V'_+$, $\dot V''_-$ with respect to time, determined from the systems (10), (8), (9):

$$\begin{aligned} \dot V_1 &= - W_1(x_1, \dots, x_n) = - \sum_{i,j=1}^{n} \beta_{ij} x_i x_j; \\ \dot V' &= - W_1(x\ , x_2, \dots, x_n); \\ \dot V'' &= - W_1(x\ , x_2, \dots, x_n). \end{aligned} \tag{22}$$

where

$$\beta_{ij} = \beta_{ji} = (\alpha_{in} a_j + \alpha_{jn} a_i) a_n^{-1} - \alpha_{j\,i+1} - \alpha_{i\,j+1}; \quad b_0 = \alpha_{i0} = \alpha_{0j} = 0. \tag{23}$$

We shall construct the Lyapunov function V for the initial system (6)-(8) by combining the functions (20) and (21) on the reversing surface $\sigma = 0$:

$$V(x_1, \ldots, x_n) = \begin{cases} V'(x_1, \ldots, x_n) - V'(0, \ldots, 0) & \text{при } \sigma \geqslant 0 \\ V''(x_1, \ldots, x_n) - V''(0, \ldots, 0) & \text{при } \sigma \leqslant 0 \end{cases}, \tag{24}$$

where we so choose V' and V" that the following equality holds:

$$V'(x_1, \ldots, x_n) = V''(x_1, \ldots, x_n) \tag{25}$$

for $\sigma(x_1, \ldots, x_n) = 0$. The relation V' = V" is equivalent to the equation $\sum_{k=1}^{n} \alpha_{1k} x_k = 0$, so that the joining condition (25) can be written in the form $\sum_{k=1}^{n} \alpha_{1k} x_k = A \sum_{k=1}^{n} b_k x_k$, i.e., we have the relation

$$\alpha_{1k} = A b_k \quad (k = 1, \ldots, n, \; A = \text{const} > 0). \tag{26}$$

The particular choice of the value of A ~~is~~ ~~un~~important for further considerations. We set

$$\alpha_{1k} = a_1 b_k \quad (A = a_1), \tag{27}$$

which simplifies somewhat the series of expressions obtained below, and eases the transition to the consideration of the case of a single null root in Eq. (4). Taking account of (20), (21), (27), we

write (24) in the form

$$V(x_1, \ldots, x_n) \quad V_1(x_1, \ldots, x_n) + 2k_p \sigma \operatorname{sgn} \sigma. \qquad (28)$$

For solutions of X(P,t) everywhere except perhaps at points of entry of X(P,t) onto the reversing surface, the time derivative dV/dt of the function (24) along X(P,t) exists and is given by the relations

$$\frac{dV}{dt} = \begin{cases} \dot{V} & \textit{for } \sigma > 0 \\ \dot{V} & \textit{for } \sigma < 0 \\ \dot{V}_0 & \textit{for } \sigma = \dot{\sigma} = 0, \ \ddot{\sigma} > 0 > \ddot{\sigma}_+ \end{cases},$$

where V_0 is the total time derivative of the function $|V|_{|\sigma|=0} = |V_1(x_1, \ldots, x_n)|_{|\sigma|=0}$ by virtue of the system (14).

It is seen from (28) that for positive definiteness of the form $V_1(x_1,\ldots,x_n)$, the regions $V(x_1,\ldots,x_n) \leq c$ = const ($c > 0$) of the space x are closed, contain the origin, and are inclosed one inside another in order of decrease of c. Making use of this fact, we can carry out a study of the stability of the system (6)-(7) in terms of the investigation of the Lyapunov function $V_1(x_1,\ldots,x_n)$ for the stable linear system (10). It is well known that if the form $W_1(x_1,\ldots,x_n)$ is positive definite, then the form $V_1(x_1,\ldots,x_n)$ is likewise (A. M. Lyapunov). However, it follows from (23), (26) and $b_n = 0$ that $\beta_{11} = 0$ and the form W_1 cannot be

sign-definite. Therefore, we shall make use of the following modification of the Lyapunov theorem.

Theorem 1. If all the roots of the characteristic equation of the system

$$dx_i/dt = \sum_{s=1}^{n} p_{is} x_s \quad (i = 1, \ldots, n) \tag{29}$$

have negative real parts, then, whatever was the sign-definite form W_1 of order m, vanishing only on the set M, and not containing positive half-trajectories of the system (29), except $O(0,\ldots,0)$, there exists one and only one form V_1 of the same order, satisfying the equation

$$dV_1/dt = \sum_{s,r=1}^{n} \frac{\partial V}{\partial x_s} p_{sr} x_r = -W_1, \tag{30}$$

and this form will necessarily be positive definite.

Actually, the single (see [12], p. 61) form V_1, satisfying (30), cannot take on negative values in any region about the origin, since the conditions of Krasovskiĭ's theorem on instability would be seen to be satisfied for the system (29). The form V_1 cannot be sign-positive since, choosing the point P_0 in this case different from 0 (for which $V_1(P_0) = 0$), and considering the positive half-trajectory $f(P_0,t)$ $(f(P_0,0) = P_0)$ of the system (29), we arrive at the equality $V_1\left[f(P_0,t)\right] = 0$ and also $W_1\left[f(P_0,t)\right] = 0$, for $t > 0$, which is inconsistent with the

absence of positive half-trajectories $f(P,t)$ on M. Consequently, the form V_1 can only be positive definite.

Since $\beta_{11} = 0$, the it is necessary for the sign-positive character of the form $W_1(x_1,\ldots,x_n)$ that

$$\beta_{1k} = 0 \qquad (k = 2, \ldots, n). \tag{31}$$

Then
$$W_1(x_1, \ldots, x_n) = W_1(x\ , x_2, \ldots, x_n) =$$

$$= W'_1(x\ , x_2, \ldots, x_n) = W_1^* = \sum_{i,j\ 2}^{n} \beta_{ij} x_i x_j, \tag{32}$$

where the coefficients β_{ij} satisfy the relations (23) and (27).

We assume that $n(n-1)/2$ of the quantities $\alpha_{ij} = \alpha_{ji}$ $(i,j = 2,\ldots,n)$ can be so chosen that (31) is satisfied, and the function W_1^* is shown to be positive definite as a form of the variables $x_1,\ldots,x_n$. In this case, W_1^* will be a sign-positive form of the variables $x_1,\ldots,x_n$, vanishing only on the M set $x_2 = \ldots = x_n = 0$, without a constant for all trajectories of the system (1?) except the trajectory $O(0,\ldots,0)$. By virtue of theorem 1, the form $V_1(x_1,\ldots,x_n)$ is positive definite.

The set M and (15) have only the single common point O so that, in accord with (32), at any point $Q(x_1^*,\ldots,x_n^*)$ of the set (15) different from O,

$$\dot{V}_1(Q) < 0; \quad \dot{V}_2(Q)\ \ 0. \tag{33}$$

We shall show that the inequality $\dot V_0(Q) < 0$ follows from (33).

Direct calculation yields:

$$\dot V_-(Q) = \sum_{r=1}^{n-1} (\partial V_1/\partial x_r)_Q\, x_{r+1} + (\partial V_1/\partial x_n)_Q\, f_n^-(Q); \qquad (34)$$

$$\dot V_+(Q) = \sum_{r=1}^{n-1} (\partial V_1/\partial x_r)_Q\, x_{r+1} + (\partial V_1/\partial x_n)_Q\, f_n^+(Q);$$

$$\dot V_0(Q) = \sum_{r=1}^{n-1} (\partial V_1/\partial x_r)_Q\, x_{r+1} + (\partial V_1/\partial x_n)_Q\, f_n^0(Q),$$

where $(\partial V_1/\partial x_r)_Q$ is the value of the partial derivative $\partial V_1/\partial x_r$ at the point Q, and $f_n^-(Q)$, $f_n^0(Q)$, $f_n^+(Q)$ are given by Eqs. (17).

We have, from (34),

$$\dot V_+(Q) - \dot V_0(Q) = (\partial V_1/\partial x_n)_Q\, [f_n^+(Q) - f_n^0(Q)];$$

$$\dot V_-(Q) - \dot V_0(Q) = (\partial V_1/\partial x_n)_Q\, [f_n^-(Q) - f_n^0(Q)],$$

whence, in accord with (16), we have

$$0 > \dot V_+(Q) \geqslant \dot V_0(Q) \geqslant \dot V_-(Q)$$

for $(\partial V_1/\partial x_n)_Q > 0$, and

$$0 > \dot V_-(Q) > \dot V_0(Q) > \dot V_+(Q)$$

for $(\partial V_1/\partial x_n)_Q > 0$.

Thus, in the case under consideration, the function $V[X(P,t)]$ does not increase along any trajectory X(P,t) while the set M, on which the derivative dV/dt exists and is equal to zero, does not contain the entire trajectories X(P,t). The essential conditions of the theorems [11] on the stability

as a whole are satisfied, inasmuch as the infinitely large function $V(x_1,\ldots,x_n) \geq 0$ is continuous and vanishes only for $x_1 = \ldots = x_n = 0$.

In consideration of the conditions of sign-definiteness of the form $W_1^* = \sum_{i,j=2}^{n} \beta_{ij} x_i x_j$, it is convenient to choose as variables not the coefficients $\alpha_{ij} = \alpha_{ji}$ $(i,j = 2,\ldots,n)$, but the nondiagonal elements β_{ij} $(i \neq j)$ of the discriminant D_{n-1} of the form W_1^*, connected with them by the nonspecial linear transformation (23). The diagonal elements β_{kk} are expressed in terms of them, in accord with (23), (27) and (31), by the relations

$$\beta_{kk} = 2\left[B_{kn} + \sum_{r=0}^{k-3} (-1)^r \beta_{k+r+1,\,k-r-1}\right] \qquad (k = 2,\ldots,n), \tag{35}$$

where

$$B_{kn} = b_{k-1} a_k - \sum_{r=0}^{k-2} (-1)^r (b_{k+r} a_{k-r-1} + b_{k-r-2} a_{k+r+1}); \tag{36}$$

$$b_{n+l} = a_{n+l+1} = a_{i,n+l} = \beta_{n+l,i} = 0 \tag{37}$$

for $l \geq 1$ and arbitrary i. Then, applying Sylvester's theorem to the form $W_1^*(x_2,\ldots,x_n)$, we formulate the following proposition:

Theorem 2. If the elements $\beta_{ij} = \beta_{ji}$ $(i \neq j)$ of the determinant D_{n-1}, whose diagonal elements satisfy the relations (35)-(37), can be so chosen that all the successive diagonal minors of this determinant are positive, then the relay system (6)-(7), which satisfies assumptions a), b) and conditions (5)

is asymptotically stable as a whole.

We note that, for the investigation of relay systems (6)-(7) which satisfy conditions a) and (5), in the case of a neutral linear part ($a_1 = 0$), we can employ the Lyapunov functions obtained from (28) and (27) for $a_1 \to 0$. Starting from theorem 1, we can easily show that the stability criteria, set up by means of this function, coincide with the corresponding criteria obtained with theorem 2, if we set $a_1 = 0$ in the latter. This should be kept in mind in applying theorem 2 to specific relay systems.

6. Theorem 2 is convenient for setting up simple stability criteria for systems of higher order.

Example 1. We can immediately find the very simple criterion

$$B_{kn} > 0 \qquad (k = 2, \ldots, n), \tag{38}$$

which contains only simple rational functions of the parameters $a_1, \ldots, a_{n-1}, b_1, \ldots b_{n-1}$, from theorem 2 by setting $\beta_{ij} = 0$ $(i \neq j)$.

For $n = 4.5$ we can, as the result of rather simple calculations, obtain the most exact of stability criteria that are obtainable by the given method.

Example 2. Let us considerthe system 1 ($n = 4$) which satisfies the restrictions a), (3), (5), and whose linear part is characterized by the transfer function

$$W(p) = \frac{b_3 p^2 + b_2 p + b_1}{a_5 p^4 + a_4 p^3 + a_3 p^2 + a_2 p + a_1},$$

and is stable or neutral. In the latter case the given set of equations can serve as a mathematical model of a relay tracking system with accelerating sections (see [1], pages 82-83, 216-221) in the calculation of electromagnetic and electromechanical time constants of the motor and the time constant of the amplifier. For example, the latter must be taken into account when the error signal from the output of the potentiometer is fed to a magnetic amplifier.

In the notation employed above,

$$D_3 = \begin{vmatrix} 2B_{2.4} & \beta_{23} & \beta_{24} \\ \beta_{32} & 2(B_{3.4}+\beta_{2.}) & \beta_{34} \\ \beta_{42} & \beta_{43} & 2B_{4.4} \end{vmatrix}. \qquad (39)$$

Simultaneous satisfaction of the inequalities

$$B_{2.4} > 0, \quad B_{3.4}+\beta_{24} \quad 0, \quad 4B_{4.4}B_{24} - \beta_{21}^2 > 0 \qquad (40)$$

is the necessary condition that the form $W_1^*(x_2, x_3, x_4)$ be positive definite. Setting $\beta_{23} = \beta_{34} = 0$ in the inequalities (39), we obtain

$$D_3 = 2(B_{3.4}+\beta_{24})(4B_{1,1}B_{24} - \beta_{21}^2).$$

so that the condition (40) is also sufficient. Rewriting (40) in the form

$$-2\sqrt{B_{1,4}B_{2,4}} < \beta_{24} < 2\sqrt{B_{4,4}B_{2,4}}\ ; \quad -B_{3,4} < \beta_{24},$$

we immediately obtain the necessary and sufficient conditions for the solvability of in inequalities (40) relative to β_{12} :

$$\begin{aligned} B_{2,4} &= b_1 a_2 - b_2 a_1 > 0; \\ B_{3,4} &= b_2 a_3 - b_3 a_2 - b_1 a_4 > -2\sqrt{B_{2,4}(b_3 a_1 - b_2 a_3)}. \end{aligned} \tag{41}$$

In accord with theorem 2, the conditions (41) are sufficient for the stability as a whole of the relay system under consideration.

Example 3. For n = 5,

$$D_4 = \begin{vmatrix} 2B_{2,5} & \beta_{23} & \beta_{24} & \beta_{25} \\ \beta_{32} & 2(B_{3,5}+\beta_{24}) & \beta_{34} & \beta_{35} \\ \beta_{42} & \beta_{43} & 2(B_{4,5}+\beta_{53}) & \beta_{45} \\ \beta_{52} & \beta_{53} & \beta_{54} & 2B_{5,5} \end{vmatrix}, \tag{42}$$

where, by (36), (37),

$$B_{2,5} = b_1 a_2 - b_2 a_1; \quad B_{3,5} = b_2 a_3 - b_3 a_2 + b_4 a_1 - b_1 a_4;$$
$$B_{4,5} = b_3 a_4 - b_4 a_3 - b_2 a_5 + b_1 a_6; \quad B_{5,5} = b_4 a_5 - b_3 a_6.$$

Simultaneous satisfaction of the inequalities

$$B_{2,5} > 0, \qquad B_{5,5} > 0; \tag{43}$$

$$4B_{5,5}(B_{3,5}+\beta_{24})-\beta_{53}^2>0, \qquad 4B_{2,5}(B_{4,5}+\beta_{53})-\beta_{24}^2>0 \quad (44)$$

is the necessary condition that the form $W_1^*(x_2,\dots,x_5)$ be positive definite. Setting $\beta_{23}=\beta_{25}=\beta_{34}=\beta_{45}=0$ in (42), similar to what was done for the case $n = 4$, we find that this condition is also sufficient. The inequalities in (44) can be replaced by equalities and the condition can be sought that there exist a single simple real solution of the resultant system of quadratic equations. Making use of corresponding results [4] (pages 95-97), we arrive at the following stability conditions:

$$B_{2,5}>0; \qquad B_{5,5}>0; \qquad (45)$$
$$M^3+N^3-M^2N^2-9/8MN+27/956>0,$$

where $M=0{,}25B_{2,5}^{-1/3}B_{4,5}B_{5,5}^{-2/3}$, $N=0{,}25B_{2,5}^{-2/3}B_{3,5}B_{5,5}^{-1/3}$.

7. For $n > 5$, a broad class of sufficient conditions for stability is obtained without excessive complication of the calculations if, without considering one or two of the elements β_{ij} $(i \neq j)$ identically equal to zero, we investigate inequalities similar to (40) or (44), and at the same time make more precise the criterion (38).

<u>Example 4.</u> Thus, in the case $n = 6$, to consider what is necessary for the careful description of the linear part of even

rather simple relay systems, we arrive at the set of inequalities

$$
\begin{gathered}
B_{5,6} + \beta_{16} > 0; \qquad B_{3,6} + \beta_{24} > 0; \\
4B_{2,6}B_{4,6} - \beta_{24}^2 > 0; \quad 4B_{4,6}B_{6,6} - \beta_{46}^2 > 0; \qquad (46) \\
4B_{2,6}B_{4,6}B_{6,6} - B_{2,6}\beta_{46}^2 - B_{6,6}\beta_{24}^2 > 0,
\end{gathered}
$$

with $\beta_{2\gamma}$ and $\beta_{\gamma 6}$ remaining as variables. By considerations similar to those given in example 2, we find the stability criterion for the whole from (46): $B_{2,6}, B_{4,6}, B_{6,6} > 0$, $B_{5,6}^{\;2}B_{2,6} + B_{6,6}B_{3,6}^{\;2} <$ $< 4B_{2,6}B_{4,6}B_{6,6}$.

Without taking in into account that $\beta_{2\gamma}$ and β_{35} are equal to zero, we have

$$
D_5 = 4B_{6,6}\begin{vmatrix} 2B_{2,6} & 0 & \beta_{24} & 0 \\ 0 & 2\,(B_{3,6} + \beta_{24}) & 0 & \beta_{35} \\ \beta_{24} & 0 & 2\,(B_{4,6} + \beta_{53}) & 0 \\ 0 & \beta_{53} & 0 & 2B_{5,6} \end{vmatrix}.
$$

Calculations similar to those given in example 3 yield an extension of the stability region (38), determined by the relations $B_{2,6}, B_{5,6}\ B_{6,6} > 0$ and (45), where

$$
M = 0{,}25B_{2,6}^{1/3}B_{3,6}B_{5,6}^{2/3}; \qquad N = 0{,}25B_{2,6}^{2/3}B_{4,6}B_{5,6}^{1/3}.
$$

The case $\beta_{16} \neq 0$, $\beta_{35} \neq 0$ is considered similarly.

The simple inequalities

$$B_{4,6} + \beta_{35} + \beta_{26} > 0; \quad 4B_{3,6}B_{5,6} - \beta_{35}^2 > 0; \quad 4B_{2,6}B_{6,6} - \beta_{26}^2 \quad 0,$$

which are obtained for β_{26} and β_{35}, lead to the following extension of the region (38):

$$B_{2,6}, B_{3,6}, B_{5,6} B_{6,6} > 0; \quad B_{4,6} > -2\left(\sqrt{B_{3,6}B_{5,6}} + \sqrt{B_{2,6}B_{6,6}}\right).$$

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INVESTIGATION OF THE STABILITY OF CERTAIN LINEAR DISTRIBUTED SYSTEMS

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(Abstract) A basis is given for carrying out investigations on the stability of distributed systems in the investigation of the roots of the characteristic equation of a linearized system. Moreover, in a large number of different examples, which themselves are of interest, the methods of construction and investigation of regions of stability of distributed systems are described.

I. DEFINITION AND CRITERION OF STABILITY

1. We shall understand by a dynamical system apparatus of any physical nature which possesses an input and an output and a uniquely defined output variable y(t) in terms of an input $x(\tau)$ for $\tau \leq t$. The set of operations which is necessary to employ on the

functions x(t) (defined for $\tau \leq t$) in order to determine y(t) is known as the operator of the system. The system is called linear if its operator is linear.

We shall call the dynamical stable if small input perturbations produce small perturbations in its output. For preciseness of this definition, it is necessary to set out some quantitative characteristic quantities of the input and output perturbations. If these numerical characteristics are denoted for the input and output by the symbols r and ρ, respectively, then the requirement of stability is that for an arbitrary $\varepsilon > 0$ $\rho y < \varepsilon$ if only $rx < \delta$, where $\delta > 0$ and depends only on ε. It follows from this definition, in particular, that for $rx \to 0$, $\rho y \to 0$.

2. We assume that the input and output variables x(t) and y(t) permit the Laplace transformation and that the connection between them realized by them in the transform can be written in the form:

$$y(p) = K(p)\,x(p). \tag{1.1}$$

As is well known, [1], if follows from the condition

$$\int_0^\infty |f(t)|^2 e^{-2\sigma t}\,dt < +\infty \tag{1.2}$$

that the transform F(p) of the function f(t) is an analytic function of p in the half-plane Re $p > \sigma$ and that for arbitrary $\sigma' > \sigma$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\sigma' + i\omega)|^2\,d\omega = \int_0^\infty |f(t)|^2\, e^{-2\sigma' t}\,dt < +\infty\,; \tag{1.3}$$

on the other hand, if F(p) is analytic in the half-plane Re $p > \sigma$

and, moreover,

$$\mathrm{Sup}_{\gamma'>\gamma} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\gamma' + i\omega)|^2 \, d\omega < +\infty \, ; \tag{1.4}$$

then F(p) has an original satisfying the condition (1.2).

By way of quantitative characteristics of r and ρ, we have the quantities

$$rf = \int_0^{\infty} |f|^2 e^{-2\gamma t} \, dt, \quad \rho f = \int_0^{\infty} |f|^2 e^{-2\gamma t} \, dt. \tag{1.5}$$

Then, in accordance with the statements formulated above, for rf = ρf the following theorem is valid:

Theorem 1. In order that a linear system be stable relative to all actions of x(t) for which $\rho x < +\infty$, it is necessary that the function K(p) be analytic for Re $p > \gamma$ and sufficient that it be analytic in some half-plane Re $p > \gamma'$, where $\gamma' < \gamma$.

Actually, on the one hand, K(p) x (p) must be an analytic function in the half-plane Re $p > \gamma$. On the other hand, if K(p) is an analytic function in the larger half-plane Re $p > \gamma'$, then for Re $p \geq \gamma$, the modulus of K(p) is bounded and the following inequality holds:

$$\rho y = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |K(\gamma + i\omega)|^2 \, |x(\gamma + i\omega)|^2 \, d\omega < \frac{M^2}{2\pi} \int_{-\infty}^{+\infty} |x(\gamma + i\omega)|^2 \, d\omega =$$

$$= M^2 \int_0^{\infty} |x(t)|^2 e^{-2\gamma t} \, dt = M^2 \rho x \, .$$

3. It does not follow from $\rho y \to 0$ that $y(t)e^{-\Gamma t} \to 0$. Therefore, the stability in the sense of the metric (1.5) with $\Gamma = \gamma = 0$ is the necessary but not the sufficient condition for the stability in the sense

of the metric usually applied: $rf = \rho f = \mathrm{Sup}_{t>0} |f(t)|$.

We assume that $K(p)$ has the original $\psi(t)$. Then, as is well known,

$$y(t) = \int_0^t \psi(t-\tau)\, x(\tau)\, d\tau,$$

and therefore the following estimate holds:

$$|y(t)|\, e^{-\Gamma t} \leqslant e^{-\Gamma t} \left(\int_0^t |\psi(\tau)|\, e^{\gamma\tau}\, d\tau \right) \mathrm{Sup}_{\tau<t} (|x(\tau)|\, e^{-\gamma\tau}), \qquad (1.6)$$

in which for $x(\tau) = \delta e^{\gamma\tau} \operatorname{sgn} \psi(t-\tau)$ the equality sign holds. The transformation of the function $\psi^*(\tau) = \psi(\tau) e^{-(\gamma-\Gamma)\tau}$ is

$$K^*(p) = \frac{p}{p-\gamma+\Gamma} K(p);$$

therefore, making use of the inequality of Bunyakovsky-Schwartz and noting that the original of $dK^*(p)/dp$ is $\tau\psi^*(\tau)$, we have:

$$\int_0^\infty |\psi^*(\tau)|\, d\tau = \int_0^\infty (1+\tau^2)^{1/2} |\psi^*(\tau)| (1+\tau^2)^{-1/2}\, d\tau \leqslant$$

$$\leqslant \left[\int_0^\infty (1+\tau^2) |\psi^*(\tau)|^2\, d\tau \int_0^\infty (1+\tau^2)^{-1}\, d\tau \right]^{1/2} =$$

$$= \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(|K^*(i\omega)|^2 + \left| \frac{dK^*}{dt} \right|^2_{p=i\omega} \right) d\omega \int_0^\infty (1+\tau^2)^{-1}\, d\tau \right]^{1/2}.$$

Then for $\gamma = \Gamma$ and $\gamma < \Gamma$, respectively, theorems 1a and 1b hold.

Theorem 1a. In order that a linear system be stable in the sense of the metric

$$rf = \mathrm{Sup}_{t>0}\, e^{-\gamma t} |f(t)|, \quad \rho f = \mathrm{Sup}_{t>0}\, e^{-\Gamma t} |f(t)| \qquad (1.7)$$

for $\Gamma = \gamma$, it is necessary that it be stable in the sense of the metric

(1.5) with $\Gamma = \gamma$, and sufficient that the function K(p) be analytic in some half-plane Re $p > \gamma'$ with $\gamma' < 0$ and that the following integral converge:

$$\int_{-\infty}^{+\infty} |dK/dp|^2_{p=i\omega}\, d\omega. \tag{1.8}$$

Theorem 1b. For stability in the sense of the metric (1.7) it is sufficient that $K^*(p)$ be analytic in the right hand plane and the integrals (1.8) and (1.4) for $K^*(p)$ converge with $\sigma = 0$.

The latter theorem for $\gamma = 0$ gives the estimate (1.6) of the speed of convergence of the transition process.

4. As a rather general example, we consider a system described by the equations

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial x} - c_1 u = f_0(u); \tag{1.9}$$

$$\frac{d\xi_i}{dt} - \sum_{s=1}^{n} a'_{is}\xi_s = f_i(\xi_1, \xi_2, \ldots, \xi_n) \quad (i = 1, 2, \ldots, n-2) \tag{1.10}$$

in which $u(0,t)$, $u_x(0,t)$, $u(1,t)$, $u_x(1,t)$ are denoted by ξ_1, ξ_2, ξ_3, ξ_4, respectively; $f_0(u)$ and $f_i(\xi_1, \xi_2, \ldots, \xi_n)$ are continuous functions of their arguments satisfying the Lipshitz condition and vanishing for $u = \xi_1 = \xi_2 = \ldots = \xi_n = 0$; a, b, c_1, a'_{is} are constants. For an investigation of the stability of a trivial solution of the system (1.9), (1.10), we consider the corresponding linearized system, adding arbitrary, bounded perturbations $\alpha(x,t)$, $\alpha_1(t)$, ..., $\alpha_{n-2}(t)$:

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + \alpha(x,t); \tag{1.11}$$

$$\frac{d\xi_i}{dt} = \sum_{s=1}^{n} a_{is}\xi_s + \alpha_i(t) \qquad (i = 1, 2, \ldots, n-2);$$

$$\xi_1 = u(0, t), \quad \xi_2 = u_x(0, t), \quad \xi_3 = u(1, t), \quad \xi_4 = u_x(1, t). \tag{1.12}$$

In transforms for zero initial conditions, the system (1.11), (1.12) has the form

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + (c - p^2)u - \alpha(x, p) = 0; \tag{1.13}$$

$$\sum_{s=1}^{n} a_{is}(p)\xi_s - \alpha_i(p) = 0; \tag{1.14}$$

$$u(0,p) = \xi_1(p), \; u_\lambda(0,p) = \xi_2(p), \; u(1,p) = \xi_3(p), \; u_x(1,p) = \xi_4(p), \tag{1.15}$$

where

$$a_{is} = (p\delta_{is} - a_{is}), \quad \delta_{is} = \begin{cases} 0, & i \neq s \\ 1, & i = s \end{cases}.$$

The solution of the system (1.13)-(1.15) in transforms can be written in the form:

$$B = K(p)\, A, \tag{1.16}$$

where $\underset{\sim}{A}$ is a vector with components $\alpha(x,p)$, $\alpha_1(p)$, $\alpha_2(p)$, ..., $\alpha_{n-2}(p)$; $\underset{\sim}{B}$ is a vector with components $u(x,p)$, $\xi_1(p)$, $\xi_2(p)$, ..., $\xi_n(p)$, and by K(p) is understood the matrix

$$\frac{1}{\Delta(p)} \left\| \begin{matrix} \int_0^1 K_0(x, \xi, p)\,d\xi, & \int_0^1 K_1(\xi, p)\,d\xi, & \ldots, & \int_0^1 K_n(\xi, p)\,d\xi \\ K_{01}(x, p) & K_{11}(p) & \ldots & K_{n1}(p) \\ K_{02}(x, p) & K_{12}(p) & \ldots & K_{n2}(p) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K_{0,n-2}(x, p) & K_{1,n-2}(p) & \ldots & K_{n,n-2}(p) \end{matrix} \right\|, \tag{1.17}$$

where

$$\Delta(p)=\begin{Vmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \lambda_1 & \lambda_2 & 0 & -1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ e^{\lambda_1} & e^{\lambda_2} & 0 & 0 & -1 & 0 & 0 & 0 & \ldots & 0 \\ \lambda_1 e^{\lambda_1} & \lambda_2 e^{\lambda_2} & 0 & 0 & 0 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} & \bar{a}_{14} & \bar{a}_{15} & \bar{a}_{16} & \ldots & \bar{a}_{1n} \\ 0 & 0 & \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} & \bar{a}_{24} & \bar{a}_{25} & \bar{a}_{26} & \ldots & \bar{a}_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & \bar{a}_{n-2,1} & \bar{a}_{n-2,2} & \bar{a}_{n-2,3} & \cdot & \cdot & \cdot & & \bar{a}_{n-2,n} \end{Vmatrix},$$

where
$$\lambda_{1,2} = -ba^{-1} \pm [b^2a^{-2} - (c-p^2)\,a^{-1}]^{1/2}.$$

Here by the product $\left(\int_0^1 K_j d\xi\right) \alpha(x,p)$ is meant $\int_0^1 K_j \alpha(\xi,p)\, d\xi$ $(j = 0, 1, 2, \ldots, n-2)$.

The vector $\underset{\sim}{A}$ can be considered as a set of input quantities of the system under investigation while the vector $\underset{\sim}{B}$ can be interpreted as the set of its output quantities. Then it is natural to call the matrix K(p) the transfer coefficient of this system from its input $\underset{\sim}{A}$ to its output $\underset{\sim}{B}$. The element $\frac{1}{\Delta} K_{ij}(p)$ of the matrix K(p) is the transfer coefficient from the ith component of the input to the jth component of the output. If $\sigma(x,p) \neq 0$, $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-2}(p) = 0$, then

$$u(x,p) = \int_0^1 \frac{K_0(x,\xi,p)}{\Delta(p)}\,\alpha(\xi,p)\,d\xi;$$

$$\xi_j(p) = \int_0^1 \frac{K_{0j}(\xi,p)}{\Delta(p)}\,\alpha(\xi,p)\,d\xi,$$

and for $\alpha_i(p) \neq 0$ and

$$
\begin{aligned}
z(x,p) = z_1(p) = \ldots = z_{i-1}(p) = z_{i+1}(p) = \ldots = z_{n-2}(p) = 0 \\
u(x,p) = K_{0i}(x,p)\,\Delta^{-1}(p)\, z_i(p); \\
\xi_j(p) = K_{ji}(p)\,\Delta^{-1}(p)\, z_i(p).
\end{aligned}
$$

The expression found above for the transfer coefficient can be a non-unique function of p. In this connection the question arises as to the separation of its unique branch, which corresponds to the desired original. This problem can easily be solved on the basis of theorem 5 of reference [1].*

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*That is, one must choose the branch of the transform whose value for $|p| \to \infty$ and $-\pi/2 + \varepsilon < \arg p < \pi/2 - \varepsilon (\varepsilon > 0)$ tends to zero.

The problem of the existence of the solution of the linearized system under consideration by means of the transform found above, in accordance with theorem 8 of reference [1], reduces to proof of the following sufficient conditions: the analytic character of the vector $\underset{\sim}{B}$ in some half-plane Re $p > \sigma$ and fulfillment for the expression of

$$
\|K(x,p)\|^2 = |\Delta(p)|^{-2} \operatorname{Sup}_{x\epsilon(0,1)} \Big\{ \sum_{i=0}^{n} \Big| \int_0^1 K_i(x,\xi,p)\,d\xi \Big|^2 + \sum_{i=1}^{n-2} |K_{0i}(x,p)|^2 +
$$

$$
+ \sum_{i=1}^{n} \sum_{j=1}^{n-2} |K_{ij}(p)|^2 \Big\};
$$

$$
\|K_x(x,p)\|^2 = |\Delta(p)|^{-2} \operatorname{Sup}_{x\epsilon(0,1)} \Big\{ \sum_{i=0}^{n} \Big| \int_0^1 K_{ix}(x,\xi,p)\,d\xi \Big|^2 + \tag{1.18}
$$

$$
+ \sum_{i=1}^{n-2} |K_{0ix}(x,p)|^2 \Big\};
$$

$$\|K_{xx}(x,p)\|^2 = |\Delta(p)|^2 \operatorname{Sup}_{x\in(0,1)} \Big\{ \sum_{l=0}^{n} \Big| \int_0^1 K_{lxx}(x,\xi,p)\,d\xi \Big|^2 + + \sum_{i=1}^{n-2} |K_{0ixx}(x,p)|^2 \Big\}$$

for Re p > σ the condition

$$\max\{\|K(x,p)\|^2,\ \|K_x(x,p)\|^2,\ \|K_{xx}(x,p)\|^2\} < c|p|^{+\alpha},$$

in which $1 > \alpha > 0$ and $C > 0$.

According to theorem 1, for stability of the initial linear system analyticity is necessary and analyticity and boundedness of the transfer coefficient K(p) in the half-plane Re p > 0 are sufficient (in particular here it follows that σ can be set equal to zero). Therefore, if $\|K(p,x)\|^2 < C$ in the vicinity of the point $|p| = \infty$, where $C > 0$, and the elements of the matrix $K(p,x)\,\Delta(p)$ are analytic functions in the half-plane Re p > 0, then the linearized system is stable, if $\Delta(p)$ does not have roots with Re p > 0, and unstable if it has even one root with Re p > 0. Thus the problem of the stability reduces to the investigation of the roots of the so-called characteristic equation

$$\Delta(p) = 0. \tag{1.19}$$

We designate the original of K(p,x) by G(t,x). Then $\overline{B}(x,t) = \int_0^t G(t-\tau,x)\overline{A}(x,\tau)\,d\tau$. We introduce the norm

$$\|G(t,x)\| = \operatorname{Sup}_{x\in(0,1)} \sum_{i=1}^{n+1}\sum_{j=1}^{n-1} |g_{ij}|,$$

where g_{ij} are the elements of the matrix G(t,x). Then, in accord-

ance with theorem 3 [5, 4], if the linearized system is stable, then for any functions $f_0(u)$, $f_i(\xi_1, \ldots, \xi_n)$ $(i = 1, 2, \ldots, n-2)$ which satisfies a Lipshitz condition for $\max \{ |u|, |\xi_1|, \ldots, |\xi_n| \} < M$ (with constants F_j $(j = 0, \ldots, n-2)$, for which the inequalities $F_j \operatorname{Sup}_{0 \leq t \leq \infty} \int_0^\infty ||G(t - \tau)|| \, d\tau < 1$ are valid), and the system (1.9), (1.10) is stable.

1. THE PROBLEM OF I. N. VOZNEZENSKIĬ ON THE STABILITY OF THE CONTROL OF A HYDRAULIC TURBINE

1. The Rauth-Hurwitz problem for quasi-polynomials arose in connection with the questions posed by I. N. Voznezenskii on the effect on the regulation of turbines of wave phenomena in the delivery conduit and the control rod [9]. A similar problem was considered in [5, 7]. The first results for quasi-polynomials were obtained by N. G. Chebotarev, after a lecture of whom on this problem a number of other mathematicians became interested. The results obtained by N. G. Chebotarev, L. S. Pontryagin [10], N. N. Meiman, B. Ya. Levin, V. N. Tsapyrin and others were set forth in detail in the well-known monograph of N. G. Chebotarev and N. N. Meiman [9]. However, these results did not appear to be sufficient for the solution of concrete problems put forth by I. N. Voznezenskii. Subsequently, partial methods were introduced for the solution of a number of the problems advanced by I. N. Voznezenskii [8, 11]. Complete solution of the problem of the effect of the delivery conduit on the regulation was given in [12, 13, 15].

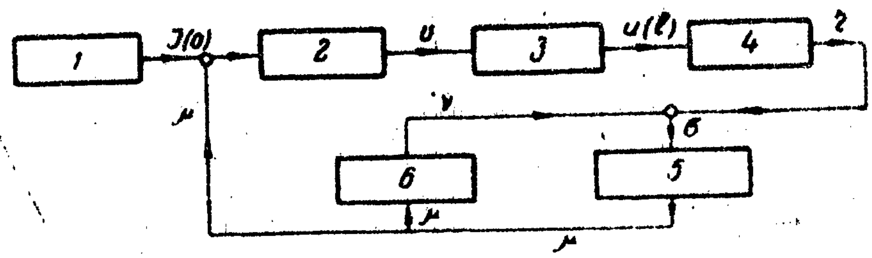

Fig. 1. Block diagram of the hydroturbine regulation.

1--pipe line, 2--hydroturbine, 3--rod, 4--regulator, 5--servo-motor, 6--isodrome.

The problem of I. N. Voznezenskii on the effect of wave phenomena in the delivery conduit and control rod on the stability of the regulation process will be considered below. Such a study makes it possible to discover a new peculiarity which can be encountered in the investigation of distributed systems. This peculiarity is connected with the character of the idealization and the role in the stability of the roots of the characteristic equation with very large negative parts.

2. The equations of motion of the system of the regulation of a hydraulic turbine with account of wave phenomena in the delivery conduit and control rod can be described in the following form (the corresponding block diagram is shown in Fig. 1): the wave equation of the control rod in relative displacements of the angular velocities with account of internal friction is

$$\frac{\partial^2 u}{\partial t^2} \quad m^2 \frac{\partial^2 v}{\partial t^2} - n^2 \frac{\partial^3 u}{\partial x^2 \partial t} = 0; \tag{2.1}$$

the condition of the equality of the angular velocities of the axes of the hydroturbine and the control rod at the place of their junction

$$u \Big|_{x=0} = v; \tag{2.1a}$$

the equality of the moments at the end of the rod

$$K \frac{\partial^2 u}{\partial t^2} \Big|_{x=l} = - GJ \frac{\partial u}{\partial x} \Big|_{x=l} - \lambda G \frac{\partial^2 u}{\partial x \, \partial t} \Big|_{x=l}; \tag{2.1b}$$

the equation of the hydroturbine

$$T_u \frac{\partial v}{\partial t} + v - \mu - \frac{3}{2}\zeta \Big|_{z=0} = 0\,; \tag{2.1c}$$

the equation of the regulator

$$T_r^2 \frac{d^2\eta_1}{dt^2} + T_k \frac{d\eta_1}{dt} + \delta\eta_1 + u \Big|_{x=l} = 0\,; \tag{2.1d}$$

the equation of the valve

$$\sigma - \eta_1 + \nu = 0; \tag{2.1e}$$

the equation of the servomotor

$$T_s \frac{d\mu}{dt} - \sigma = 0\,; \tag{2.1f}$$

the equation of the isodrome

$$T_i \frac{d\nu}{dt} + \nu - \beta T_i \frac{d\mu}{dt} - \alpha\mu = 0\,; \tag{2.1g}$$

the wave equation of the conduit

$$\frac{\partial u}{\partial t} = \frac{gh_0}{v_0}\frac{\partial \zeta}{\partial t}, \quad \frac{\partial \zeta}{\partial t} = \frac{a_1 v_0}{gh_0}\frac{\partial y}{\partial t}\,; \tag{2.1h}$$

the equation of the discharge at the end of the conduit connected to the turbine

$$\mu + (0{,}5\zeta - y)_{z=0} = 0; \tag{2.1i}$$

the condition of complete reflection of the pressure wave at the end of the conduit line L, connected to a large water reservoir

$$\zeta\big|_{z=L} = 0\,. \tag{2.1j}$$

In this case the following notation is employed: $m^2 = G^{1/2}\rho^{-1/2}$, $n^2 = \lambda^{1/2}\rho^{-1/2}$, G is the shear modulus, ρ is the density of the rod material, J is the polar moment of the transverse cross section of

the rod, K is the moment of inertia of the connecting mass of the regulator, λ is the coefficient of internal friction of the rod, $\tau_2 = lm^{-1}$ = the time of flight of a wave of the rod of length l. The sense of the remaining parameters and notation are clear from reference [12].

First let us look at the case in which the delivery conduit (short pipe) and damping in the rod are not taken into account. In this case the transform of the system (2.1) for zero initial conditions (with consideration of the perturbations $\alpha_1(t), \ldots, \alpha_7(t)$, their right hand parts in the notations $\eta = \xi_5$, $\nu = \xi_7$) can be written in the form:

$$m^2 \frac{d^2u}{dx^2} - p^2u = 0; \tag{2.2}$$

$$\begin{aligned}
&p\xi_1 - \xi_8 = \alpha_1(p);\\
&p\xi_2 - \xi_1 = \alpha_2(p);\\
&GJ\xi_3 - Kp\xi_1 = \alpha_3(p);\\
&p\xi_5 - \xi_6 = \alpha_4(p);\\
&\xi_8 + \delta\xi_5 + (pT_r^2 + T_k)\xi_6 = \alpha_5(p);\\
&T_s\alpha\xi_1 - \beta T_i\xi_5 + [T_sT_ip + (T_s + \beta T_i)]\xi_7 - T_sT_u\alpha\xi_8 = \alpha_6(p);\\
&T_s\xi_1 - \xi_5 + \xi_7 + T_sT_up\xi_8 = \alpha_7(p);
\end{aligned} \tag{2.3}$$

$$u|_{x=0} = \xi_1(p), \quad u|_{x=1} = \xi_2(p), \quad u_x|_{x=1} = \xi_3(p). \tag{2.4}$$

Solving the system (2.2)-(2.4), we find $\underset{\sim}{B}(u, \xi_1, \ldots, \xi_8)$ with components

$$u(x, p) = \sum_{j=1}^{7} \frac{\Delta_{1,j+3}e^{px/m} + \Delta_{2,j+3}e^{-px/m}}{\Delta}\alpha_j(p); \tag{2.5}$$

$$\xi_i(p) = \sum_{j=1}^{7} \frac{\Delta_{i+2,\,j+3}}{\Delta}\sigma_j(p) \qquad (i = 1,2,\ldots,8); \tag{2.6}$$

$$\Delta(p) = \begin{vmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{p\tau_1} & e^{-p\tau_1} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{p}{m}e^{p\tau_1} & -\frac{p}{m}e^{-p\tau_1} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & p & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & GJ & -Kp & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \delta & T_r^* p + T_k^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta T_i & 0 & d & -\sigma T_s T_a \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & pT_sT_a \end{vmatrix} \tag{2.7}$$

Here $d = T_i T_s p + T_s + \beta T_i$. From the form of the determinant (2.7) of the system (2.3), (2.4), it follows that their highest order is determined by the principal term $p^7 e^{p\tau_1}$, while its expansion in the elements of the fifth column shows that the order of its minors $\Delta_{ij}(p)$ cannot be higher than this order. By virtue of this fact, for Re $p > \gamma > 0$ the estimate

$$\operatorname{Sup}_{x\in(0,1)}\{|u(x,p)|, |\xi_1|, \ldots, |\xi_8|\} \leq Cp^{-1}, \tag{2.8}$$

takes place, from which, according to Sec. 1 of the present paper, it follows that the study of the stability of the system (2.1) for $\tau_1 = 0$ (τ_1 is the time of flight of the shock wave of the delivery conduits) reduces to consideration of the characteristic equation $\Delta(p) = 0$, which in our case has the form:

$$P_5(p)\,(\sigma_2 p \operatorname{sh} p\tau_2 + \operatorname{ch} p\tau_2) + P_1(p) = 0, \qquad (2.9)$$

where

$$P_1(p) = T_i p + 1, \quad P_5(p) - (T_a p + 1)\,(T_r^2 p^2 + T_k p + \delta) \times$$
$$\times\,|T_i T_s p^2 + (T_s + \beta T_i)\,p + \alpha|, \quad \tau_2 = K/J\sqrt{G\varrho}.$$

In the construction of the D cut of the plane of the parameters σ_2, τ_2, we use the method set forth in [13]. For this purpose, solving (2.9) relative to the parameter σ_2 and equating its imaginary part equal to zero (after the substitution $p = i\omega$, $-\infty < \omega < +\infty$), we find the values

$$\omega_j = +\left[-\frac{p}{2} + \left(\frac{p^2}{4} - q\right)^{1/2}\right]^{1/2},$$

where
$$p = \frac{\alpha(T_a T_r^2 - T_i T_r^2 - T_i T_k T_a) + \delta(T_r^2 + T_k T_a - T_i T_a)T_i -}{T_i^2 |T_s T_k T_a + T_r^2(T_s + \beta T_a)|}$$
$$\frac{\delta T_i^2 T_s - \delta\beta T_i^2 T_a + T_s(T_k T_a + T_r^2)}{T_i^2 |T_s T_k T_a + T_r^2(T_s + \beta T_a)|};$$

$$q = \frac{\delta(T_s + \beta T_i) - \alpha T_k - \alpha\delta(T_a - T_i)}{T_i^2 |T_s T_k T_a + T_r^2 (T_s + \beta T_a)|},$$

for which

$$\sigma_2 = a \operatorname{ctg} \omega\tau_2 + b \csc \omega\tau_2 \qquad (2.10)$$

takes on real values ($a = \omega^{-1}$, $b = P_5^{-1}(i\omega)(1 + T_i\omega i)\omega^{-1}$).

For clarification of the rule of the shading of the curve of the D cut of the plane of the parameters σ_2, τ_2 and the determination of sgn $b(\omega_j^*)$, we construct the auxiliary curve $W = P_1(i\omega)/P_5(i\omega)$, the form of which, for the most interesting case in practice, $q < 0$

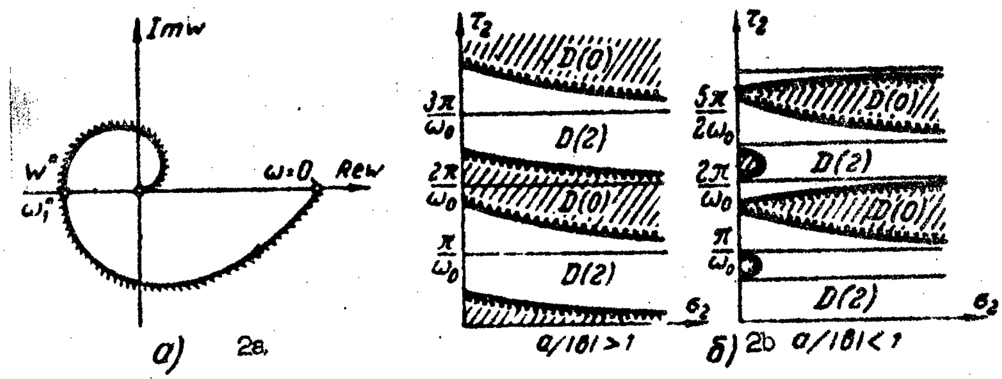

Fig. 2. D-cut of the plane of parameters

[14], is given in Fig. 2a. We note from Fig. 2a that $b(\omega_1^*) < 0$. The boundary curve of the D cut of the plane of the parameters σ_2, τ_2 is, according to (2.10), the sum of the curves $\sigma_2' = \omega_1^{*-1}$ ctg $\omega_1^* \tau_2$ and $\sigma_2'' = b(i\omega_1^*) \csc \omega_1^* \tau_2$ for the same values of τ_2; its form (see Fig. 2b) depends essentially on the behavior of $a|b|^{-1}$. The direction of the shading of the boundary curve is determined by the change in the sign of Im $\sigma_2 = (\omega^2 - \omega_1^{*2}) \sin^{-1}(\omega \tau_2)$ in the vicinity of ω_1^*. In the transition through the boundary curve in the plane σ_2, τ_2 along the line $\tau = \tau_0$, $\tau_0 \in (s2\pi\omega_1^{*-1}, (2s+1)\pi\omega_1^{*-1})$ from left to right in the characteristic equation (2.9) in the plane p two roots with Re $p > 0$ are lost (but for $\tau_0 \in ((2s+1)\pi\omega_1^{*-1}, 2(s+1)\pi\omega_1^{*-1})$ are gained).

We shall show that the claimed stability regions, shaded in Fig. 2b, are actually regions of instability. For this purpose, we consider the quasi-polynomial $P_5(p) + P_1(p)$ corresponding to the point $\sigma_2 = \tau_2 = 0$. From the D cut of the plane W of the quasi-polynomials $WP_5(p) + P_1(p)$ (see Fig. 2a) it follows that for $|W^*| = a|b|^{-1} \geq 1$, the polynomial $P_5 + P_1$ does not have a single root to the right of the negative axis (since the point $W = \infty$ corresponds to the polynomial $P_5(p)$, which does not have a single root to the right of the imaginary axis), while for $a|b|^{-1} \leq 1$, the polynomial $P_5 + P_1$ has two roots with Re $p > 0$.

Proceeding in similar fashion, we separate the regions of instability in the planes of the parameters δ, T_r^2. After substitution

in the expression for δ obtained from (1.9), $p = i\omega(-\infty \leq \omega < +\infty)$ we have:

$$\delta = \omega^2 T_r^2 - T_k \omega i - P_1(i\omega) P_3^{-1}(i\omega) Q(i\omega), \qquad (2.11)$$

where

$$P_3(i\omega) = P_1(i\omega)[\alpha - T_s T_i \omega^2 + (T_s + \beta T_i)\omega i], \; Q(i\omega) = (-\omega\sigma_2 \sin\omega\tau_2 + \cos\omega\tau_2)^{-1}.$$

The values of ω_j for which δ becomes real are found from the condition of intersection of the curve $z_1 = -\mathrm{Im}\,[P_1(i\omega)P_3^{-1}(i\omega)Q(i\omega)]$ with the line $z_2 = T_k\omega$ (Fig. 3a). In the case considered of the boundaries, the D cuts are the straight lines

$$\delta = \omega_j^{*} T_r^2 + \delta_{0j}, \qquad (2.12)$$

where

$$\delta_{0j} = z_3(i\omega_j^{*}) = -\mathrm{Re}\, P_1(i\omega_j^{*}) P_3^{-1}(i\omega_j^{*})\, Q(i\omega_j^{*}).$$

The direction of the shaded lines (2.12) is determined by the sign of $\mathrm{Im}\ \delta$ in the region of ω_j. The D cut of the plane of the parameters δ, T_r^2 for z_1, z_2, z_3, which are shown in Fig. 3a, b, is given in Fig. 3c. It is seen from Fig. 3a, b, that as $T_k \to 0$, $\omega_j^{*} \to \infty$, and $\delta_{0j} \to 0$, i.e., the system becomes everywhere unstable. On the other hand, with increase of T_k the regions of stability grow. There is no point in going into detail on the separation of the regions of instability in the space $T_i T_s$, $T_s + \beta T_i$, α, since it is performed in similar fashion.

We now proceed to the consideration of the general case. In accord with what was set forth in Sec. 1, the investigation of the instability of the system (2.1) reduces to a consideration of the

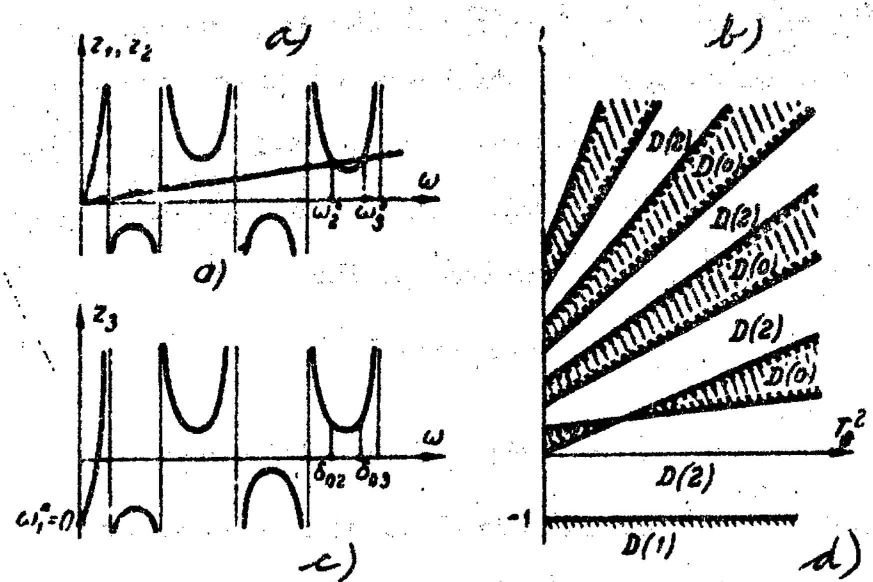

Fig. 3. D-cut of the plane of parameters T_r^2, σ ($\tau_1 = \lambda = 0$).

characteristic equation

$$P_5(p)(\sigma_2'\gamma\,\mathrm{sh}\,\gamma l+\mathrm{ch}\,\gamma l)(2\sigma_1\,\mathrm{ch}\,p\tau_1+\mathrm{sh}\,p\tau_1)+ \tag{2.13}$$
$$+2(\sigma_1\,\mathrm{ch}\,p\tau_1+\mathrm{sh}\,p\tau_1)P_1(p)=0,$$

where $\sigma_2=KJ^{-1}$, $\sigma_1=gh_0v_0^{-1}\beta^{1}$, $\gamma=\pm p\rho^{1/2}(G+\lambda p)^{1/2}$.

For a comparison of the equation of the rod and conduit the natural damping in the control rod is taken into account. From the physical point of view it is natural to assume that a very small damping ~~in the~~ control rod cannot have an essential effect on the control process. However, fuller consideration of this problem shows that the neglect of arbitrarily small damping leads to the impossibility of satisfaction of the condition of stability.

We attempt to separate the regions of stability in the general case for $\lambda = 0$, for example, in the plane of the parameters σ_1, τ_1. Proceeding in similar fashion to [13], we arrive at the equation of the boundary curves of the D cut

$$\sigma_1=\xi_j^*\,\mathrm{ctg}\,\omega_j^*\tau_1$$

of the plane of the parameters σ_1, τ_1. The values of ξ_j^* and ω_j^* correspond to the points of intersection of the circle $f = -(2i+\xi)/(\xi - i)$ with the curve $g = [P_5(i\omega)/P_1(i\omega)]Q(i\omega)$ (see Fig. 4). It follows directly from Fig. 4 that as $\omega \to \infty$, $\omega_j^* \to \infty$, while the values $\mu_j \to 0$, correspondingly. In this case the curve of the D cut fills the entire plane of the parameters σ_1, τ_1. This latter signifies

that in the idealized system assumed, without damping in the control rod, it is always unstable, and that this instability takes place at very high frequencies.

In the consideration of arbitrarily small damping in the control rod, the possibility is established of separating the region of stability. We show this in the example of the separation of the regions of stability in the plane of the parameter $\delta_1 T_r^2$. Solving the characteristic equation (2.13) relative to the parameter δ, we have:

$$\delta = \omega^2 T_r^2 - T_k \omega i - f(i\omega), \tag{2.14}$$

where

$$f(i\omega) = \frac{2\sigma_1 \cos\omega\tau_1 - 2i \sin\omega\tau_1 \, P_1(i\omega)}{2\sigma_1 \cos\omega\tau_1 + i \sin\omega\tau_1 \, P_3(i\omega)} Q(i\omega).$$

The form of the curve $f(i\omega)$ is shown in Fig. 5a. After construction of the curve $\delta'(i\omega) = -T_k \omega i + f(i\omega)$ (Fig. 5b), which is simultaneously the curve of the D cut of the plane of the complex parameter δ for $T_r^2 = 0$, it is easy to proceed to the desired D cut. For this purpose, we construct a series of straight lines in the plane $\delta_1 T_r^2$

$$\delta = \omega_j^{*2} T_r^2 + \delta_{0j} \qquad (\delta_{0j} = \operatorname{Re} \delta'(i\omega_j^*)),$$

the number of which is finite in the case under discussion. The final form of the D cut of the plane of the parameters δ, T_r^2 is given in Fig. 5c (the shading on the line is in accord with Fig. 5b).

We shall investigate how the curve bounding the stability region changes with change of T_k from 0 to ∞. For $T_k = 0$, curve b) goes over into curve a) of Fig. 5, from which it is seen that $\omega = \omega_j^*$

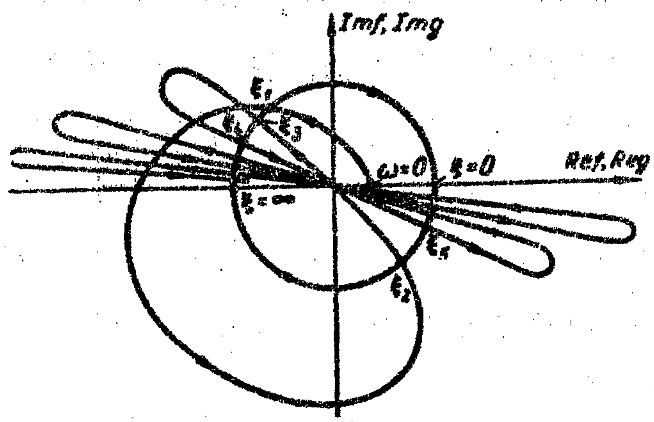

Fig. 4.

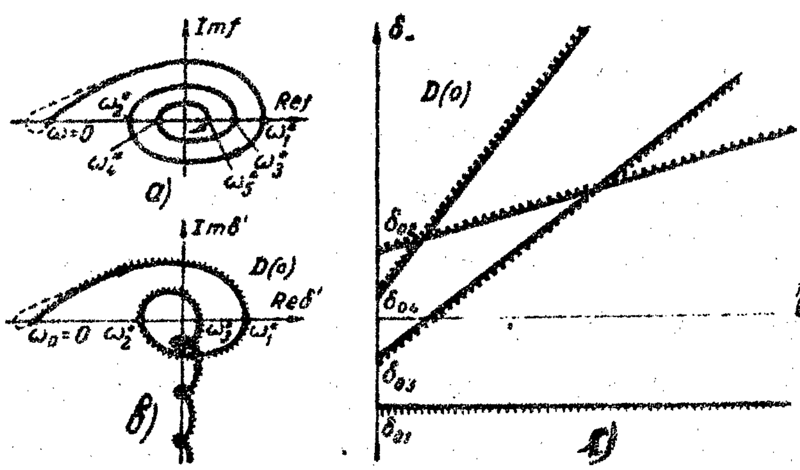

Fig. 5. D-cut of the plane of parameters T_r^2, $\delta(\tau_1 \neq 0, \lambda \neq 0)$.

takes on a denumerable set of values going off to infinity with increase in j. Simultaneously, δ_{0j}, taking positive and negative values, tends to zero. In the δ, T_r^2 plane, the region of stability here is extended to the halfplane of stability $\delta > \delta_*$, while the remaining part of the plane $\delta > 0, T_r^2 \geq 0$ becomes unstable. With increase in T_k the lines on the δT_r^2 plane with large slopes disappear and the region of stability is expanded.

For some value $T_k = T_{k1}$, a single line remains with $\delta > 0$ with the region of stability higher than this straight line. With further increase of T_k, the value of ω_0 decreases and $\delta_0 \to 0$; thereafter δ_0 becomes less than zero and tends to -1 for

$$T_k \to \frac{3\tau_1 + \alpha\delta T_a + \alpha T_k + \delta (T_s + \beta T_i) - \alpha\delta T_i}{\alpha\delta}.$$

The straight line of the D-cut $\delta = \omega_1 T_r^2 + \delta_{01}$ joins with the line $\delta = -1$, and the system becomes stable everywhere.

It is interesting to note the effect of T_k on the frequency of excitation of the system. For $T_k = 0$, the system is excited over the entire spectrum of frequencies ω_j, which represents a denumerable set. For $T_k \neq 0$, the infinite spectrum of frequencies falls out, and with further increase of T_k, the system is excited over all very low frequencies.

3. INVESTIGATION OF THE CONDITIONS OF SELF-EXCITATION WITH DELAYED FEEDBACK POSSESSING DISPERSION

Let us consider an ideal amplifier with delayed feedback

in the circuit.* If we denote by J(p) the transfer coefficient of the

*This problem was considered in reference [17].

feedback, then the characteristic expression of the system under consideration has the form

$$1 - kJ(p) = 0. \tag{3.1}$$

We shall consider below the case in which the feedback is realized by a length of line and LC is the circuit with account and without account of dissipation.

1. In the first case, the characteristic equation has the form

$$k = \exp\left|x\sqrt{(LCp^2+RG)+(LG+RC)p}\right|,$$

where x is the distance from the beginning of the line; R, G, L, C, are the resistance, leakage, inductance and capacity per unit length of line. As a parameter, in terms of which we shall investigate the stability, we choose the amplification coefficient k which we shall consider complex during the course of the investigation. As usual, setting $p = i\omega$, we obtain the equation of the boundary of the D-cut for the parameter k. One can remark that for $\omega \to \infty$ the boundary curve wraps itself asymptotically around the circle. We shall explain how this comes about. Introducing the notation

$$\sqrt{RG - LC\omega^2 + (LG+RC)\omega i} = \sqrt{x+iy} = \sqrt{0,5(x+\sqrt{x^2+y^2})} +$$

$$+ i\sqrt{0,5(\sqrt{x^2+y^2}-x)} = m + in,$$

and as the result of simple calculations, we obtain the result that as $\omega \to \infty$

$$m^2 = 0,5y^2\left(\sqrt{x^2+y^2}-x\right)^{-1} \to (LG+RC)^2(4LC)^{-1},$$

that is,

$$m \to m_\infty = (LG+RC)\,0,5\,(LC)^{-1/2}$$

as $\omega \to \infty$. Moreover, $dm/d\omega > 0$ for $\omega > 0$ and $\sqrt{RG} < (LG + RC) \times 0.5(LC)^{-1/2}$. By virtue of this, the boundary curve fits around the circle of radius e^{α/m_∞} (Fig. 6a) from the inside, and therefore the system is stable for values of k lying in the interval $[k(\omega_1), k(0)]$, where $|k(\omega_1)| > |k(0)|$.

2. Now let the feedback be realized by a matched semi-infinite LC chain (see [18]), where $z_1 = Lp + R$, $z_2 = (Cp + G)^{-1}$.*

---

*In reference [17] only the case $R = G = 0$ is investigated.

We denote the current in the nth section by i_n. Then, from the equation for the nth section,

$$(z_1 + 2z_2)i_n - z_2(i_{n+1} + i_{n-1}) = 0 \qquad (3.2)$$

we find an expression for the current i_n. We shall seek this expression in the form $i_n = Ae^{-n\lambda}$, since the chain is matched and, consequently, there is no reflected wave. Let $z_2 i_0 = u$; then we obtain the result that $i_n = uz_2^{-1}e^{-n\lambda}$ and the transfer coefficient of the

feedback $J(p) = e^{-n\lambda}$. Moreover, after the substitution $i_n = Ae^{-n\lambda}$ in (3.2), we get

$$e^{-\lambda} + e^{\lambda} = 2 + z_1 z_2^{-1}.$$

Then, in accord with the rule for selection of the branch, formulated on page 244,

$$e^{-\lambda} = 1 + 0{,}5\, z_1 z_2^{-1} - \sqrt{(1 + 0{,}5\, z_1 z_2^{-1})^2 - 1},$$

and therefore the transfer coefficient can be written in the form:

$$J(p) = (\alpha - \sqrt{\alpha^2 - 1})^n; \quad \alpha = 1 + 0{,}5\,(Lp + R)\,(Cp + G).$$

If we denote $\alpha + \sqrt{\alpha^2 - 1}$ by η, then the characteristic equation takes the form

$$k = \eta^n. \tag{3.3}$$

Evidently this equation has n complex roots η_i $(i = 1, 2, \ldots, n)$, lying on a curve of radius $\sqrt[2n]{|k|}$. We choose the quantity $\sqrt[2n]{|k|}$ as the parameter in terms of which we shall investigate the stability.

If the dissipative parameters R and G are not taken into account, then the hodograph $\eta(i\omega)$ has the form shown in Fig. 6b. If the amplification coefficient is smaller than unity in absolute value, then all the roots of the characteristic equation have a negative real part and the system in this case will be stable. If now $|k| > 1$, then some of the η_i will be found outside the hodograph and the roots of the characteristic equation corresponding to them will have a positive real part, that is, the system will be unstable.

In the case of consideration of the dissipative parameters R and G, the hodograph $\eta(i\omega)$ is shown in Fig. 6c. For all ω, $|\eta| > 1$.

Actually, carrying out a series of transformations, we obtain the result that

$$\begin{aligned} 4|\eta| = \sqrt{x^2+y^2} + \sqrt{(x+4)^2+y^2} + \\ + \sqrt{(x+\sqrt{x^2+y^2})(x+4+\sqrt{(x+4)^2+y^2})} + \\ + \sqrt{(\sqrt{x^2+y^2}-x)(\sqrt{(x+4)^2+y^2}-x-4)}. \end{aligned} \tag{3.4}$$

It is not difficult to see that $|\eta|$ can be equal to unity only for $\omega = 0$, but $\eta(0) = 1 + 0.5RG + \sqrt{0.25R^2G^2 + RG} > 1$. Now let $|\eta| = a$. We then obtain from (3.4) a biquadratic equation in ω:

$$(a^2 - 1)^4 + 4a^4x^2 - 4a^2(a^2 - 1)^2x = a^2(a^2 + 1)(x^2 + y^2),$$

which has no more than two positive roots. Thus the hodograph $\eta(i\omega)$ for $\omega > 0$ has no more than two points to which corresponds the one and the same value $|\eta|$. Consequently, the critical values of the amplification coefficient k, for which change in the stability of the system takes place, will generally be different for different signs of k.*

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*These results do not coincide with the results of reference [17], where it is confirmed that the change in the stability of the system takes place at a definite value of $|k|$, i.e., the critical value of k does not depend on the sign of k.

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4. INVESTIGATION OF THE STABILITY OF CONTROL OF AUTOMATIC COMPRESSOR STATION GAS SUPPLY WITH ACCOUNT OF WAVE PHENOMENA IN INPUT AND OUTPUT GAS MAINS

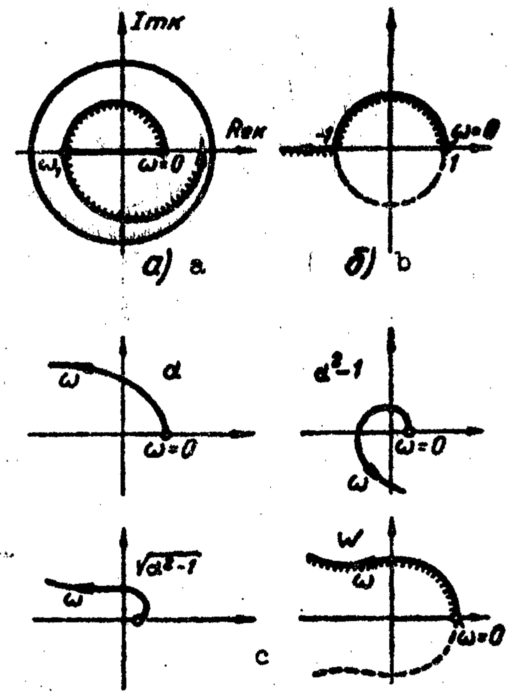


Fig. 6. D-cut in the parameters k and η. Figure labels are a, b, c.

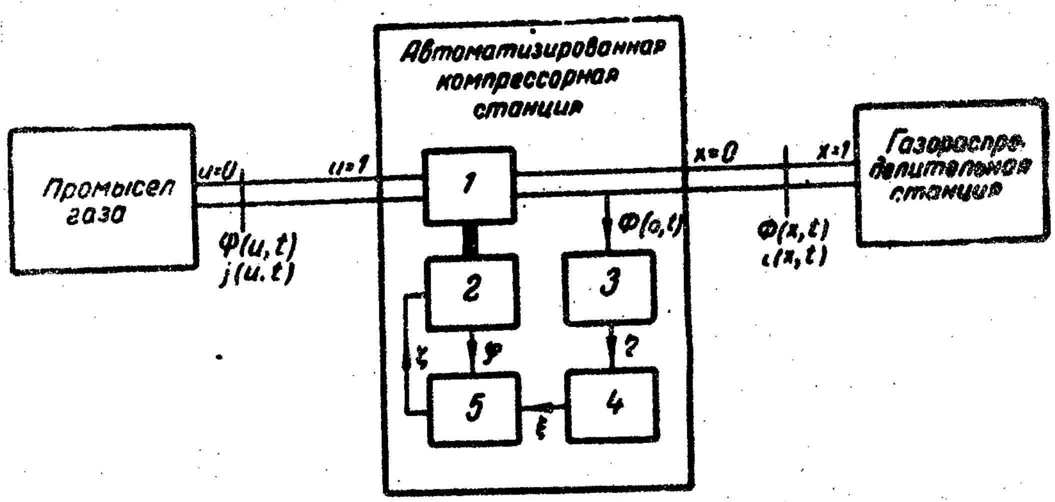


Fig. 7. Block diagram of automatic compressor station: 1--compressor, 2--gas motor, 3--pneumatic isodrome regulator, 4--servomotor, 5--centrifugal velocity regulator. (In drawing, on the left: gas supply, in the middle: automatic compressor station, on the right: gas distribution station.)

1. The introduction of new techniques which guarantee an increase in the productivity of compressor stations in the transport along long pipe lines of liquids and gases has led to a very unpleasant phenomenon, connected with the vibrations of the main pipe lines. The struggle with vibrations at the present time has acquired great importance; however, in the majority of cases it is carried on blindly and requires great material efforts [24].

A block diagram of an automatic compressor station is given in Fig. 7. From the source, which is a large gas reservoir, the gas travels through a long gas pipe to the automatic compressor station, from which it discharges into a gas distribution station along a second delivery conduit. In Fig. 7, i(x, t) and j(x, t) denote the relative variations of delivery of gas into the input and output gas mains, while Ψ (x, t) and Φ (x, t) are the relative pressure changes.

The operation of the automatic compressor station is described by the following set of equations:

for the output of the compressor station [19]

$$j = \alpha_1 \Psi - \alpha_2 \Phi + \alpha_3 \varphi ; \qquad (4.1)$$

the equation of the gas motor compressor

$$T_a \dot{\varphi} + \varphi = \alpha_4 \Psi + \alpha_5 \zeta - \alpha_6 \Phi ; \qquad (4.1a)$$

the equation of the centrifugal regulator of velocity with corrector

$$T_r^2 \ddot{\zeta} + T_k \dot{\zeta} + \delta_1 \zeta = \beta \xi - \varphi ; \qquad (4.1b)$$

the equation of the membrane-lever mechanism

$$\xi = \eta; \tag{4.1c}$$

the equation isodrome pneumatic regulator (for $T_{v_1} = T_{v_2} = 0$ [21])

$$(\delta + B)\dot{\eta} + T_u^{-1}\Phi + BT_u^{-1}\eta + \dot{\Phi} = 0. \tag{4.1d}$$

The vibrations of the gas in the input and output pipes are described by equations with partial derivatives:

$$-\frac{\partial \Psi}{\partial u} = c_1 j; \quad -\frac{\partial \Psi}{\partial t} = c_2 \frac{\partial j}{\partial u} \tag{4.2}$$

(input gas main);

$$-\frac{\partial \Phi}{\partial x} = c_1 i; \quad -\frac{\partial \Phi}{\partial t} = c_2 \frac{\partial i}{\partial x} \tag{4.3}$$

(output gas main).

Here a_1, a_2, a_3 are coefficients which are proportional to the tangents of the angles of inclination of the characteristics which determine the yield of the compressor station as a function of the pressure at the input and output, and of the angular velocity of the shaft of the gas motor compressor; similarly, a_4, a_5, a_6 are coefficients proportional to the tangents of the slope angles of the characteristics of the gas motor compressor at the points of linearization. The parameter β_1 characterizes the stiffness of the spring of the velocity regulator and the membrane lever apparatus;

$$B = a_1 r_1^{-1} i_y^{-1} [b_0 + \delta_s a_2/(r_1 + r_2)], \tag{4.4}$$

where $a_1, a_2, b_0, b_s, r_1, i_y, r_2$ are the parameters of the isodrome pneumatic regulator [21]. The meaning of the parameters T_a, T_r^2, T_k, T_u, δ, δ_1, c_1, c_2 are well understood and need no further com-

ment.

The purpose of the research was the separation of the regions of stability of the system in the plane of the parameters δ, T_u^{-1}, which characterize the isodrome pneumatic regulator under fulfillment of the following boundary conditions:*

---

*In such an arrangement, for given numerical values of the parameters c_1/c_2, $A_1, A_2 = 0$, A_3, the problem was suggested to us in 1957 by our colleague of the Kiev Institute of Gas M. A. Zhidkov. A particular case of this problem was considered in reference [20].

---

1) The condition of constancy of pressure at the end of the input gas main which is connected to the large gas reservoir

$$\Psi(0, t) = 0; \tag{4.5}$$

2) The condition of equality of the discharge of gas at the input and output of the compressor station

$$j(1, t) = i(0, t); \tag{4.6}$$

3) The condition of operation of the compressor station under the assumption that $T_r^2 = T_a = T_k = B = 0$,

$$\left.\frac{\partial i}{\partial t}\right|_{x=0} = A_1 \Psi_t\Big|_{u=1} - \left[A_2 + A_3 \frac{1}{\delta}\right] \Phi_t\Big|_{x=0} - A_3 T_u^{-1} \delta^{-1} \Phi(0, t); \tag{4.7}$$

4) The condition of constancy of the discharge of gas at the end of the output gas main

$$i(1, t) = 0. \tag{4.8}$$

In equation (4.7)

$$A_1 = \frac{\delta(\alpha_1 \alpha_5 + \alpha_3\alpha_4) + \alpha_1\alpha_5}{\delta + \alpha_5}; \quad A_2 = \frac{\delta\alpha_2 + \alpha_2\alpha_5 + \alpha_3\alpha_6}{\delta + \alpha_5};$$

$$A_3 = \frac{\delta\alpha_3\alpha_5}{\delta + \alpha_5}.$$

Applying a Laplace transformation in the set of equations (4.2)-(4.8), we obtain a characteristic equation of the following form after the usual transformations:

$$p\,\mathrm{sh}^2\gamma + \mu p\gamma\,\mathrm{ch}\,\gamma\,\mathrm{sh}\,\gamma + (\tau p + \nu)\,\mathrm{ch}^2\gamma = 0 \qquad (4.9)$$

with parameters

$$\tau = \frac{A_2}{A_1} + \frac{A_3}{A_1\delta}, \quad \nu = \frac{A_3}{\delta A_1 T_u}; \quad \mu = c_1^{-1}A_1^{-1}; \quad \gamma = c_1^{1/2}c_2^{-1/2}p^{1/2}.$$

2. We construct a D-cut of the plane of the parameters (δ, T_u^{-1}) [14]. Direct construction of the boundary of the D-cut in terms of the parameter δT_u^{-1} of interest to us leads to difficulties, connected with its behavior for $\omega \to \infty$, which is characterized by the principal terms

$$\delta p \sqrt{c_1 p/c_2}\,\mathrm{sh}\,(2\sqrt{c_1 p/c_2}). \qquad (4.10)$$

In order to avoid this difficulty, we first construct an auxiliary D-cut according to the parameters τ, ν; δT^{-1}.

For the construction of the D-cut of the plane of the parameters τ, ν, we substitute in the characteristic equation (4.9) $p = i\omega$ $(-\infty < \omega < +\infty)$. Since the left side of (4.9) is an even function relative to $\sqrt{c_1 p/c_2}$, then in the extraction of the root from i it suffices to consider only its positive value. Construction of the curve of the

D-cut one must change the value of ω from 0 to ∞, since in the change $-\infty < \omega < 0$, the curve of the D-cut of the plane τ, ν is run over a second time, which leads to double shading. After substitution in (4.9) of the root $\sqrt{c_1\omega_i/c_2} = \Omega + i\Omega$ ($\Omega = \sqrt{r\omega c_1/c_2/2}$) and division of its left hand side into real and imaginary parts, we arrive at a set of linear equations relative to the parameters τ, ν. Solving the system, we find:

$$\tau = \frac{\mathrm{sh}^2\Omega - \sin^2\Omega + \mu\Omega(\cos\Omega + \mathrm{ch}\,\Omega)(\mathrm{sh}\,\Omega - \sin\Omega)}{(\cos\Omega + \mathrm{ch}\,\Omega)^2}; \qquad (4.11)$$

$$\nu = \omega\frac{\mu\Omega(\mathrm{sh}\,\Omega + \sin\Omega)(\mathrm{ch}\,\Omega + \cos\Omega) + 2\,\mathrm{sh}\,\Omega\sin\Omega}{(\cos\Omega + \mathrm{ch}\,\Omega)^2}; \qquad (4.12)$$

$$\Delta = -\omega(\cos\Omega + \mathrm{ch}\,\Omega)^2. \qquad (4.13)$$

For $\omega = 0$ we have the fundamental straight line $\nu = 0$. The D-cut of the plane of the parameters τ, ν is shown in Fig. 8a. For investigation of the behavior of the curve of the D-cut at infinity, we consider the reflection of the semicircle $\mathrm{Re}\, p > 0$, $|p| > R$ on the plane τ, ν. Substituting $p = re^{i\phi}$ in (4.9) and separating the real and imaginary parts, we have, for sufficiently large $R > 0$:

$$\tau_\infty = \mu\sqrt{\frac{c_1 r}{c_2}}\,\frac{(1 - 4\cos^2\varphi/2)\,\mathrm{sgn}(\cos\varphi/2)}{\cos\varphi/2};$$

$$\nu_\infty = \mu\sqrt{\frac{c_1 r}{c_2}}\,\frac{\mathrm{sgn}(\cos\varphi/2)}{\cos\varphi/2}. \qquad (4.14)$$

We denote by Δ p the right semicircle $\mathrm{Re}\, p > 0$, $|p| > R$ of the point $p = \infty$, and by $\Delta\, S_p$ the reflection of this semicircle on the

plane τ, ν. Upon change of ϕ in the limits from $-\pi/2$ to $\pi/2$, ν_{∞} and τ_{∞} do not change sign ($\nu > 0$, $\tau < 0$); therefore, closure of the set ΔS_p for $p \to \infty$ contracts to the point S_{∞} [16].

It is easy to go from the D-cut of the plane of parameters ν, τ (see Figs. 8a, b, c) to the D-cut of the plane of parameters δ, T_u^{-1}, and from it to the plane δ, T_u^{-1}. The basic straight line $\nu = 0$ for the plane ν, τ breaks down in the plane τ, T_u^{-1} into two straight lines: $\tau = 0$, $T_u^{-1} \neq 0$ and $T_u^{-1} = 0$, $\tau \neq 0$.

Qualitatively, one can obtain a picture of the D-cut of the plane of the parameters τ, ν, from the D-cut of the plane of the complex parameter τ (Fig. 8d):

$$\tau = -(\operatorname{th}\Omega + \mu\Omega)\operatorname{th}\Omega - \nu/p. \tag{4.15}$$

It follows from Fig. 8d that for $\nu < 0$ the set is unstable, for $\nu > 0$ a section of stability appears, for $\nu > 0$ the section of stability increases with increase in ν.

5. ON REGIONS OF STABILITY OF AN ISODROME TEMPERATURE REGULATOR

Let us consider isodrome regulation of a heating furnace with a concentrated heat capacity inside and with heat transfer through the wall. The regulation system contains a distributed section (as a consequence of account of propagation of heat through the thickness of the wall as part of the system). Investigation of the system with a real furnace meets with great difficulty; therefore we shall consider the problem for a spherical furnace. Description of

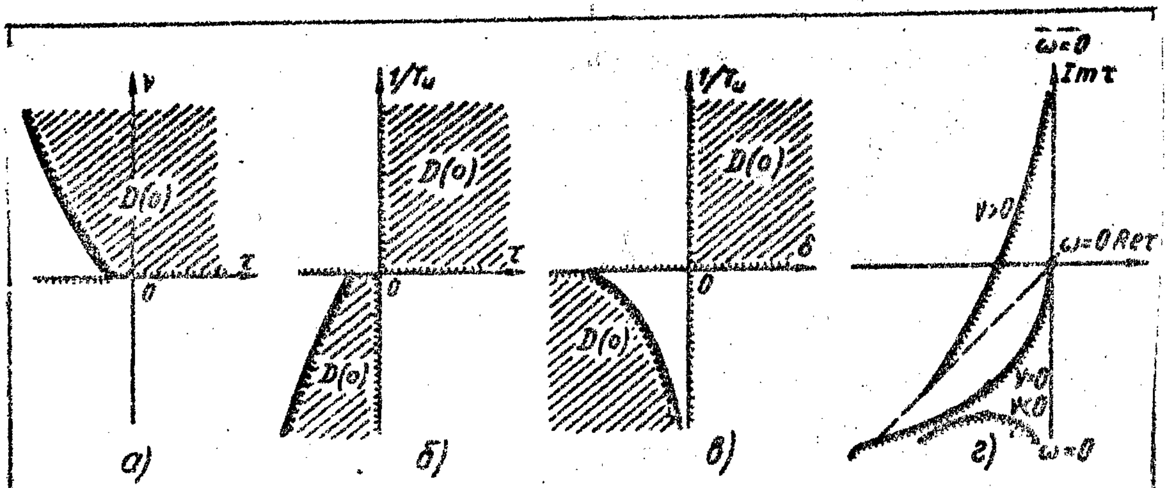


Fig. 8. The D-cut of the plane of parameters δ, T_u^{-1}.

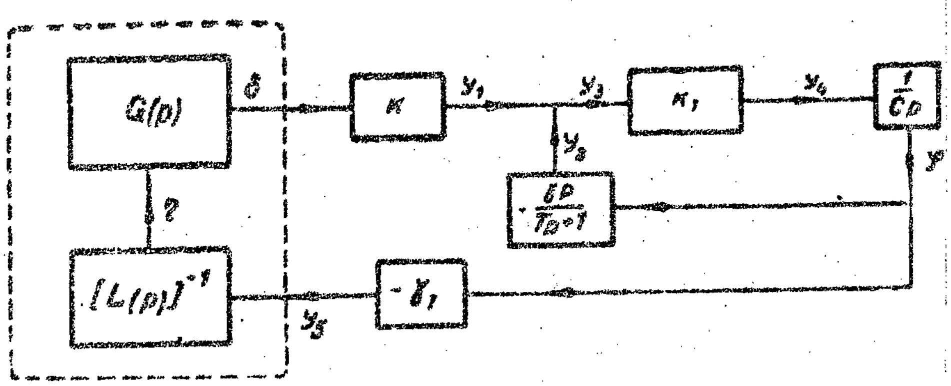


Fig. 9. Block diagram of temperature regulation.

the circuit of an isodrome regulator (Fig. 9) is given in [22]. A compensated bridge with an arm mounted in the wall served as the measuring element in the system.

To obtain the characteristic equation we first write down the equation of the furnace and the heat transfer in the wall of the furnace.

a) The equation of the furnace. We use the notation : ρ is the heat capacity of the heated body, k_2 is the coefficient of loss of power by radiation, λ is the coefficient of thermal conductivity of the material of the wall of the furnace, r_0 is the internal radius of the furnace wall, r_1 is the external radius of the furnace wall, Θ (r,t) is the temperature of the wall of the furnace, Θ_2 is the temperature of the surrounding medium, Θ_1 is the temperature inside the furnace. The equation of balance of energy for the electric furnace has the form:

$$I^2 R dt = \rho\, d\Theta_1 + k_2 \Theta_1\, dt - \lambda \frac{\partial \Theta}{\partial r}\Big|_{r_0} S_0 dt, \tag{5.1}$$

where $S_0 = 4\pi r_0^2$, R is the resistance of the heater, I is the current in the heater. Let I change by the quantity $y_5 (I \to I + y_5)$. Then $\Theta \to \Theta_1 + \eta_1$, $\Theta \to \Theta + \eta$ and we get from (5.1):

$$2IRy_5 = \rho \frac{d\eta_1}{dt} + k_2 \eta_1 - \lambda \frac{\partial \eta}{\partial r}\Big|_{r_0} S_0. \tag{5.2}$$

Let

$$\left.\frac{\partial \eta}{dr}\right|_{r_0} = H(p)\,\eta_0(p).$$

Then

$$y_3(p) = L(p)\,\eta_0(p),$$

where

$$L(p) = k_3 + \rho_1 p - \lambda_1 H(p), \tag{5.3}$$

(k_3, ρ, λ_1 are, respectively, k_2, ρ, λS_0 divided by 2IR).

For a flame furnace the energy balance equation has the form:

$$W dt = \rho\, d\Theta_1 - \lambda \left.\frac{\partial \Theta}{\partial r}\right|_{r_0} S_0 dt + k_2 \Theta_1 dt + W_1 dt, \tag{5.4}$$

where W is the energy generated in the burning of the fuel, W_1 is the energy leaving with the burning products. Let $W \rightarrow W + \Delta W$; then $\Theta_1 \rightarrow \Theta_1 + \eta_1$, $\Theta \rightarrow \Theta + \eta$, $W_1 \rightarrow W_1 + \Delta W$. One can assume [23] that $\Delta W_1 = n \Delta W$. Moreover, if $\Delta W = b y_5$, then we obtain (5.3) from (5.4), where k_3, ρ_1, λ_1 are, respectively, $k_2, \rho, \lambda S_0$ divided by $b(1 - n)$.

b) The equation of heat transfer in the wall of the furnace. Let s be the coefficient of temperature conductivity, a_1 the coefficient of heat emission at the exterior wall of the furnace, a_0 the coefficient of heat emission at the interior wall of the furnace.

The equation of heat transfer has the form:

$$\frac{\partial \Theta}{\partial t} = s \left[\frac{\partial^2 \Theta}{dr^2} + \frac{2}{r}\frac{\partial \Theta}{\partial r}\right]$$

or

$$\frac{\partial [r\Theta]}{\partial t} = s\,\frac{\partial^2 [r\Theta]}{\partial r^2}, \tag{5.5}$$

where $r_0 < r < r_1$, $t > 0$.

Boundary conditions:

$$\Theta(r,\ 0) = \Theta_0 = \Theta_2;$$

$$-\frac{\partial\Theta}{\partial r}\bigg|_{r_0} + L_0\,[\Theta(r_0) - \Theta_1] = 0; \tag{5.6}$$

$$\frac{\partial\Theta}{\partial r}\bigg|_{r_1} + L_1\,[\Theta(r_1) - \Theta_2] = 0, \tag{5.7}$$

where $L_i = \alpha_i \lambda^{-1}$.

In these representations, equation (5.5) takes the form:

$$s\,[r\Theta]_{rr} - rp\,[\Theta - \Theta_0] = 0, \tag{5.8}$$

while (5.6) and (5.7) remain without change if the representation is designated by the same letters as in the original. Solution of equation (5.8) is

$$\Theta(r,\ p) = \Theta_0 + r^{-1}\,(Ae^{qr} + Be^{-qr}),$$

where $q = \sqrt{ps^{-1}}$.

Determining A and B from (5.6) and (5.7), we obtain:

$$\Theta = \Theta_0 + \Pi\,[A_1 e^{q(r_1-r)} - B_1 e^{-q(r_1-r)}]\,(\Theta_1 - \Theta_2), \tag{5.9}$$

where

$$\Pi = L_0 r_0^2 r^{-1}\,[A_1 B_0 e^{q(r_1-r_0)} - A_0 B_1 e^{-q(r_1-r_0)}]^{-1};$$

$$A_0 = 1 + L_0 r_0 - r_0 q;\quad B_0 = 1 + L_0 r_0 + r_0 q;$$

$$A_1 = -1 + L_1 r_1 + r_1 q;\quad B_1 = -1 + L_1 r_1 - r_1 q.$$

Now let $\Theta \to \Theta + \eta\ (p)$. Then $\Theta_1 \to \Theta_1 + \eta_1$ and, from (5.9),

$$\eta_1(p) = G(p)\,\eta_0(p),$$

where

$$G(p) = \Pi\,[A_1 e^{q(r_1-r)} - B_1 e^{-q(r_1-r)}].$$

From (5.6)

$$\left.\frac{d\eta}{dr}\right|_{r_0} = L_0\,[G(r_0) - 1]\,\eta_0,$$

that is,
$$H(p) = L_0\left\{-1 + L_0 r_0\left[-1 + q r_0 + \right.\right.$$
$$\left.\left. + 2qr_0\left(\frac{1 - L_1 r_1 + r_1 q}{1 - L_1 r_1 - r_1 q}\ e^{-2q(r_1-r_0)} - 1\right)^{-1}\right]^{-1}\right\}^{-1}.$$

The characteristic equation of the system has the form:

$$\mu + (Tp + 1)^{-1} + \nu G(p) p^{-1} L^{-1}(p) = 0, \tag{5.10}$$

where $\mu = ck_1^{-1}\gamma^{-1}$, $\nu = k\gamma_1\gamma^{-1}$.

Let us consider the case of an infinitely thick wall ($l_1 = \infty$):

$$G(p) = \frac{L_0 r_0^2}{r}\ \frac{e^{-q(r-r_0)}}{1 + L_0 r_0 + r_0 q},\quad H(p) = L_0\left(\frac{L_0 r_0}{1 + L_0 r_0 + r_0 q} - 1\right).$$

By virtue of the fact that, for conjugate arguments, the functions entering into (5.10) are also conjugate, one can construct the D-cut of the plane of the parameters μ, ν by setting $p = i\omega$ and varying ω from 0 to ∞. We write down (5.10) in the form:

$$\mu + x + iy + \nu(u + iz) = 0. \tag{5.11}$$

Then

$$\mu = -x + yuz^{-1};\quad \nu = -yz^{-1},$$

$$x = (1 + T^2\omega^2)^{-1}; \quad y = -T\omega(1 + T^2\omega^2)^{-1};$$

where

$$u = -\frac{L_0 r_0^2 e^{-\psi}}{r\omega(\xi^2 + \zeta^2)}(\xi\sin\psi + \zeta\cos\psi);$$

$$z = \frac{L_0 r_0^2 e^{-\psi}}{r\omega(\xi^2 + \zeta^2)}(\zeta\sin\psi - \xi\cos\psi);$$

$$\psi = (r - r_0)\sqrt{\omega/2s};$$

$$\xi = k_3(1 + L_0 r_0 + r_0\sqrt{\omega/2s}) - \rho_1 r_0\omega\sqrt{\omega/2s} + \lambda_1 L_0(1 + r_0\sqrt{\omega/2s})\,4\pi r_0^2;$$

$$\zeta - \rho_1\omega(1 + L_0 r_0 + r_0\sqrt{\omega/2s}) + k_3 r_0\sqrt{\omega/2s} + \lambda_1 L_0 r_0\sqrt{\omega/2s}\,4\pi r_0^2.$$

For $\omega > 0$, $y < 0$ and, therefore, sgn ν = sgn z. Graphs of μ (ω) and $\nu(\omega)$ are shown in Fig. 10a, the D-cut of the plane of these parameters in Fig. 10b.

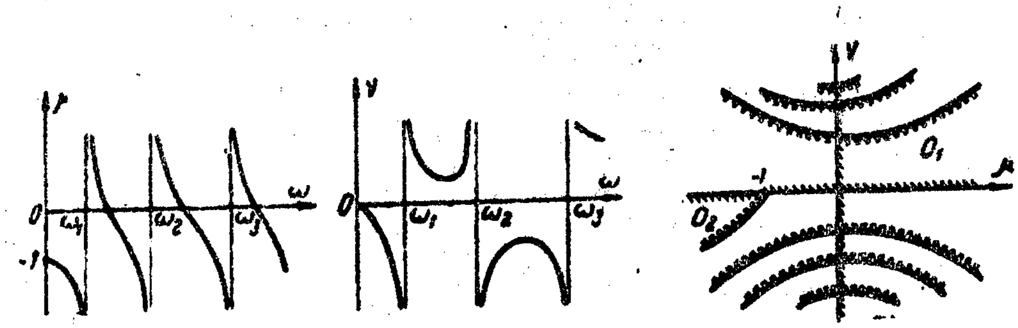
The rule for shading is determined by the sign of the expression

$$\Delta = \begin{vmatrix} 1 & u \\ 0 & z \end{vmatrix} = z.$$

If $\Delta > 0$, then the shading is to the left for motion along the boundary line on the side of increase of ω; if $\Delta < 0$, then to the right. Inasmuch as sgn Δ = sgnz = sgn ν, then for $\nu < 0$, the shading is to the right and for $\nu > 0$ to the left. The lines $\mu = 0$ and $\nu = 0$ special.

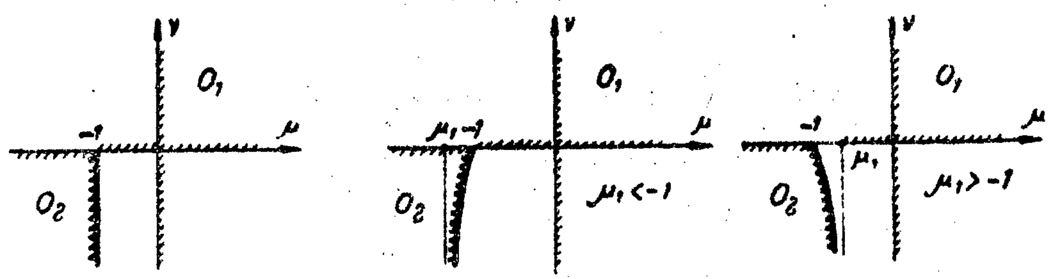
It is necessary to show that O_1 and O_2 are regions of D(0). Let $k_3 \to \infty$ or (which leads to the same result) r_0, $r \to \infty$. This means that leakage absorbs all the energy, i.e., the system will operate stably.

Actually, in this case, the characteristic equation of the



ν
ω
0
ω_1
ω_2
ω_3
-1
ν
ω
0
ω_1
ω_2
ω_3
ν
O_1
μ
-1
O_2

Fig. 10.



ν
O_1
μ
-1
O_2
ν
O_1
μ
μ_1 -1
O_2
$\mu_1 < -1$
ν
O_1
μ
-1
μ_1
O_2
$\mu_1 > -1$

Fig. 11.

system

$$\mu + (Tp + 1)^{-1} = 0$$

has the single root $p = -(1 + \mu)(T\mu)^{-1} < 0$ for $\mu > 0$ and $\mu < -1$. In this case $|\nu| \to \infty$ and the D-cut takes the form shown in Fig. 11a. For $(\mu, \nu) \in O_1$, O_2, the characteristic equation does not have roots with $\mathrm{Re}\, p > 0$, i.e., O_1 and O_2 are regions of D(0).

We note that for the model in which the resistance thermometer measures the temperature of the heated body directly, and in the case in which the thermal conductivity of the wall can be neglected (i.e., the distributed element is not taken into account and $\eta = \eta_1$), the characteristic equation of the system

$$\mu + \nu p^{-1} L^{-1}(p) + (Tp + 1)^{-1} = 0,$$

where $L(p) = \rho_1 p + k_3$, gives the following conditions for stability, with the aid of the theorems of Rauth-Herwitz:

$$\mu\nu > 0, \quad T(\alpha + \beta T\mu)\nu > \beta(1 + \mu)(\alpha + \alpha\mu + \beta T\mu).$$

These conditions are satisfied in the regions O_1 and O_2 (Fig. 11b), when $\mu_1 = -\alpha\beta^{-1}T^{-1}$.

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THE NUMERICAL METHOD OF FINDING PERIODIC MOTIONS IN AUTOMATIC CONTROL SYSTEMS

Yu. I. Neymark

(Abstract) A numerical method is considered for finding the periodic motions of an automatic control system with a single nonlinear characteristic.

The theoretical analysis of the dynamics of many systems of automatic control reduces to finding their stable periodic motions. Significant success has been achieved in finding the periodic motions of systems close to linear (the method of small parameters [1-3], the method of averaging [4-7]), for systems which satisfy the hypothesis of autoresonance or of a filter (the method of harmonic balance [8-11]), for piecewise linear systems whose investigation can be reduced to a point transformation of a straight line into a straight line [12-15], and for the simplest regime of a relay system [16-18]. However, all of these methods themselves yield only various more or less simple ways of setting up the equations of periodic motions and not ways of finding them in practice.

In the present paper a numerical method is put forth for finding the periodic motions which is applicable to an automatic control system with a single nonlinear characteristic.

1. PRELIMINARY DESCRIPTION OF THE METHOD

Let us consider an automatic control system consisting of a linear section with transfer coefficient K(p) and a nonlinear element which brings about the connection

$$y = \Omega\ (x). \tag{1.1}$$

between the input variable x and the output variable y. In the presence of an external periodic influence of frequency ω we bring it to the input of the nonlinear element and expand it in a Fourier series:

$$f(t) = \sum_k \left[\alpha_k \sin\ k\,\omega\,(t - t_0) + \beta_k \cos\ k\,\omega\,(t - t_0)\right]. \tag{1.2}$$

We assume that in the system under consideration a periodic regime of frequency ω is possible and we represent the input x to the nonlinear element in the form:

$$x = \sum_k \left[(A_k + \alpha_k)\ \sin k\,\omega\,(t - t_0) + (B_k + \beta_k)\ \cos k\,\omega\,(t - t_0)\right]. \tag{1.3}$$

In this case the output of the nonlinear element will also be a periodic function of the time ω and therefore it can be represented in the form:

$$y = \sum_k \left[\bar{A}_k \sin k\,\omega\,(t - t_0) + \bar{B}_k \cos k\,\omega\,(t - t_0)\right]. \tag{1.4}$$

where the Fourier coefficients $\overline{A}_k$ and $\overline{B}_k$ can, by virtue of (1.1) be

represented in terms of the quantities $A_1 + \alpha_1$, $A_2 + \alpha_2$, . . . and $B_0 + \beta_0$, $B_1 + \beta_1$, $B_2 + \beta_2$, . . . :

$$\overline{A}_k = \overline{A}_k\,(B_0 + \beta_0,\ A_1 + \alpha_1, \ldots);\quad B_k = \overline{B}_k\,(B_0 + \beta_0,\ A_1 + \alpha_1, \ldots). \tag{1.5}$$

On the other hand, inasmuch as y is the input of a linear chain with coefficients of transfer K(p), and x is its output, then

$$A_s = K_1\,(s\omega)\,\overline{A}_s - K_2\,(s\omega)\,\overline{B}_s;\quad B_s = K_1\,(s\omega)\,B_s + K_2\,(s\omega)\,A_s, \tag{1.6}$$

where $K_1(s\omega)$ and $K_2(s\omega)$ are the real and imaginary parts, respectively, of $K(is\omega)$.

The infinite set of equations (1.5) and (1.6) determines all the possible periodic motions. However, finding its solution presents very great difficulties, not to speak of the difficulty of actually finding Eq. (1.5).

The widely known method of harmonic balance is essentially an approximate method of solution of this set of equations. According to this method, of all the infinite set of equations (1.4) and (1.6), only the first two equations are chosen, and the remainder are replaced by the equations $A_2 = \overline{A}_2 = \ldots = 0$, $B_2 = \overline{B}_2 = B_3 = \ldots = 0$. Account of the further harmonics of the desired periodic motion in this method leads to very complicated calculations and is little applied in practice.

In the present work a new method of approximation of the infinite set of equations (1.5)-(1.6) is put forth which is based, on the one hand, on the possibility of neglecting the higher harmonics

(for which it would be sufficient to assume that for very large ω, $K_1(\omega)$ and $K_2(\omega)$ are equal to zero), and on the other hand, on a certain approximation of the nonlinear equation (1.5). This approximation, in the case in which the nonlinear character of (1.1) is piecewise linear, replaces Eq. (1.5) by a piecewise linear system. In the same way the problem of finding the periodic motions in the case of forced vibrations leads to the solution, generally speaking, of several sets of inhomogeneous linear equations, and in the case of free vibrations, to the problem of finding the frequencies ω for which a certain redefined set of linear equations has a solution and then to finding this solution.

2. APPROXIMATION OF EQUATIONS (1.5)

We divide the period $T = 2\pi/\omega$ into m equal parts; let

$$x_j = \sum_k \left[A_k \sin(kj\, 2\pi/m) + B_k \cos(kj\, 2\pi/m) + f_j\right] \tag{2.1}$$

be the input values of the nonlinear chain at the points $t_j = t_0 + jT/m$ ($j = 0, 1, \ldots, m-1$). The output values of the nonlinear chain at these same points will be $\Omega(x_0), \Omega(x_1), \ldots, \Omega(x_{m-1})$, respectively. By the well-known method of least squares, the Fourier coefficients $\overline{A}_k$ and $\overline{B}_k$ can be found approximately from the values of y at m points:[21]

$$\overline{A}_k = \frac{2}{m} \sum_{j=0}^{m-1} \Omega(x_j) \sin\left(kj \frac{2\pi}{m}\right); \tag{2.2}$$

$$\overline{B}_k = \frac{2}{m}\sum_{j=0}^{m-1} \Omega(x_j)\cos\left(kj\frac{2\pi}{m}\right). \qquad (2.2)$$

After substitution of (2.1) into (2.2), we obtain the desired approximation of Eqs. (1.5).

In finding the periodic motions, one must form equations either for the amplitudes of the harmonics A_k, B_k, or for the quantities x_0, x_1, x_2, . . ., x_{m-1}. Keeping in mind the second possibility, we find the value of the output of the nonlinear chain at the moments of time t_0, t_1, . . ., t_{m-1}:

$$y_j = \Omega(x_j) = \sum_{k=0}^{n}\left[\overline{A}_k \sin\left(kj\frac{2\pi}{m}\right) + \overline{B}_k \cos\left(kj\frac{2\pi}{m}\right)\right]. \qquad (2.3)$$

Substitution of (2.2) into (2.3) leads to approximation of Eqs. (1.5) in the case in which the quantities x_0, x_1, . . . , x_{m-1} are taken to be the fundamental unknowns.

3. THE CONSTRUCTION OF APPROXIMATE SOLUTIONS OF PERIODIC MOTIONS

In the construction of approximate solutions of periodic motions, we can take for our consideration both the number of considered harmonics n and the number of points of separation m, observing, however, the condition $m \geq 2n + 1$ in this case. By virtue of this condition, the number of unknown x_j is not smaller than the number of unknowns A_k and B_k. However, this circumstance itself cannot serve as the decisive argument in the construction of the equations relative to the unknowns A_k and B_k, inasmuch as verifica-

tion of the practicality of additional conditions is more easily carried out over the quantities x_j.

a) The equation of periodic motions in the variables A_k, B_k. Expressing the unknowns x_j in Eqs. (2.2) according to (2.1), and then eliminating the quantities $\bar{A}_k$ and $\bar{B}_k$ in accordance with (1.6), we obtain a set of 2n + 1 equations of the form:

$$A_k = \frac{2}{m}\sum_{j=0}^{m-1} \Omega \left[\sum_{s=0}^{n} A_s \sin\left(sj\frac{2\pi}{m}\right) + B_s \cos\left(sj\frac{2\pi}{m}\right) + f_j \right] \times$$
$$\times \left[K_1(k\omega)\sin\left(kj\frac{2\pi}{m}\right) + K_2(k\omega)\cos\left(kj\frac{2\pi}{m}\right)\right]; \qquad (3.1)$$
$$B_k = \frac{2}{m}\sum_{j=0}^{m-1} \Omega \left[\sum_{s=0}^{n} A_s \sin\left(sj\frac{2\pi}{m}\right) + B_s \cos\left(sj\frac{2\pi}{m}\right) + f_j \right] \times$$
$$\times \left[K_2(k\omega)\sin\left(kj\frac{2\pi}{m}\right) + K_1(k\omega)\cos\left(kj\frac{2\pi}{m}\right)\right].$$

If the system under consideration is non-autonomous ($f_j \neq 0$), then ω is known and the equations (3.1) make it possible to find 2n + 1 unknowns A_k, B_k. If the system is autonomous ($f_j = 0$), then by virtue of the arbitrariness of the selection of the starting time (or, what amounts to the same thing, of the quantity t_0) any of the quantities A_k, B_k ($k \neq 0$) can be taken as equal to zero. If it is known that the variable x certainly takes on the value x^* in its variation, then it can also be assumed that

$$x_j = \sum_{k=0}^{n} \left[A_k \sin\left(kj\frac{2\pi}{m}\right) + B_k \cos\left(kj\frac{2\pi}{m}\right)\right] - x^*. \qquad (3.2)$$

Equation (3.1), together with (3.2) or any other equation, for

example, $B_1 = 0$, gives the complete set of equations for finding this desired $2n + 2$ unknowns A_k, B_k and ω.

b) The equation of periodic motions in the variables x_j.

Expressing the unknowns A_k, B_k according to (1.6) in the relations (2.1), and then substituting in place of $\bar{A}_k$, $\bar{B}_k$ their expressions (2.2), we arrive at the following set of equations:

$$x_j = \frac{2}{m}\left\{\sum_{\sigma=0}^{m-1} \Omega(x_\sigma) \sum_{k=0}^{n}\left[K_1(k\omega)\cos\left(k(\sigma-j)\frac{2\pi}{m}\right) + K_2(k\omega)\sin\left(k(\sigma-j)\frac{2\pi}{m}\right)\right]\right\} + f_j, \tag{3.3}$$

which can be written in the form

$$x_j = \frac{2}{m}\sum_{\sigma=0}^{m-1} \Omega(x_\sigma) \sum_{s=0}^{n} \mathrm{Re}\, K(is\omega)\, e^{is(j-\sigma)2\pi/m} + f_j. \tag{3.4}$$

The number of these equations is equal to the number of unknowns $x_0, x_1, \ldots, x_{m-1}$; therefore, in the case of forced vibrations of this system, it is sufficient for finding the periodic motions; in the case of free vibrations, when ω is unknown, it is necessary to add an additional requirement to Eqs. (3.3) of the form $x_j = x^*$ or of the form $B_k = 0$, where B_k is naturally assumed to be expressed in terms of the unknowns $x_0, \ldots, x_{m-1}$.

4. CONCRETE EXAMPLE

Let the frequency characteristic $K(i\omega)$ of a linear neutral stable chain (one root with zero and the others with negative real parts) be given by the graph Fig. 1a, while the nonlinear chain has

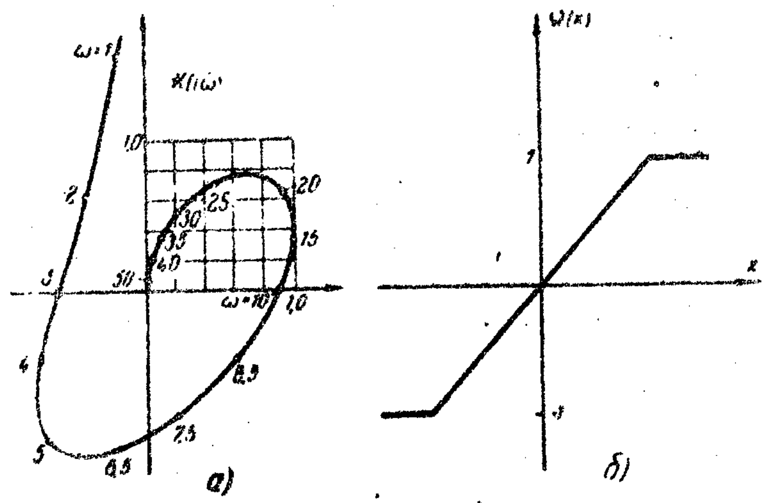

Fig. 1.

the symmetric characteristic shown in Fig. 1b, with the angular coefficient of the average part $\varkappa$.

The equilibrium state of the system under consideration will be stable for $0 < \varkappa < 0.9$. For $\varkappa = 0.9$, vibrations are excited with a frequency $\omega = 10$. From the form of the linear characteristics, it follows that in the transition through the critical value 0.9, one can expect "soft" excitations of symmetric auto-oscillations with an amplitude greater than unity.

We carry out a numerical calculation of the form and frequency of the periodic motion for $\varkappa = 1.3$.* We shall start out

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*This numerical calculation was carried out by N. S. Sukochev.

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from the Eqs. (3.3), in which we set m = 12, and the number of harmonics under consideration is n = 5. Inasmuch as we are seeking the symmetric periodic motion, then we can assume that $x_0 = x_6 = 0$ and $x_{i+6} = -x_i$ (for i = 1, . . . , 5).

The set of equations (3.3) in our case is written in the form:

$$x_j = \frac{1}{6} \sum_{s=1}^{5} \sum_{k=0}^{5} \Omega(x_s) \left\{ K_1(k\omega) \left[\cos\left(k(s-j)\frac{\pi}{6}\right) - \cos\left(k(s-j)\frac{\pi}{6} + k\pi\right) \right] + K_2(k\omega) \left[\sin\left(k(s-j)\frac{\pi}{6}\right) - \sin\left(k(s-j)\frac{\pi}{6} + k\pi\right) \right] \right\} =$$

$$= \frac{1}{3} \sum_{s=1}^{5} \Omega(x_s) \sum_{k=1,3,5} \left[K_1(k\omega) \cos\left(k(s-j)\frac{\pi}{6}\right) + K_2(k\omega) \sin\left(k(s-j)\frac{\pi}{6}\right) \right] = 0 \quad (j = 0, 1, \ldots, 5). \tag{4.1}$$

The even harmonics fall out, as was to be expected.

We assume that x_1 and x_5 are less than $1/\varkappa \simeq 0.77$, and that x_2, x_3, x_4 are greater than $1/\varkappa$; then $\Omega(x_1) = \varkappa x_1$, $\Omega(x_5) = \varkappa x_5$ and $\Omega(x_2) = \Omega(x_3) = \Omega(x_4) = 1$. In this connection, we construct the set of equations (4.1) for $\Omega = 10$ and $\Omega = 9.5$:

$$\left.\begin{array}{llllr}
3{,}4659\,x_1 & & & -\,0{,}6319\,x_5 & -\,1{,}1224 = 0 \\
-3{,}036\,x_1 & & & -\,0{,}8206\,x_5 & +\,2{,}1748 = 0 \\
0{,}6319\,x_1 & -6\,x_2 & & -\,1{,}144\,x_5 & +\,5{,}3348 = 0 \\
0{,}8206\,x_1 & \quad 6\,x_3 & & +\,0{,}5053\,x_5 & +\,5{,}4322 = 0 \\
1{,}144\,x_1 & & -\,6\,x_4 & +\,3{,}4659\,x_5 & +\,3{,}3973 = 0 \\
0{,}5053\,x_1 & & & -\,3{,}036\,x_5 & +\,1{,}1224 = 0
\end{array}\right\} \quad (4.2)$$

$$\left.\begin{array}{llllr}
3{,}3923\,x_1 & & & -1{,}2603\,x_5 & -\,1{,}8220 = 0 \\
-3{,}01\,x_1 & & & -1{,}0153\,x_5 & +\,1{,}5684 = 0 \\
1{,}2603\,x_1 & -6\,x_2 & & -1{,}387\,x_5 & +\,4{,}9284 = 0 \\
1{,}0153\,x_1 & -\,6\,x_3 & & -0{,}0246\,x_5 & +\,5{,}8790 = 0 \\
1{,}387\,x_1 & & -\,6\,x_4 & +\,3{,}3923\,x_5 & +\,4{,}0505 = 0 \\
0{,}0246\,x_1 & & & -3{,}01\,x_5 & +\,1{,}8220 = 0
\end{array}\right\} \quad (4.3)$$

Each of the set of equations (4.1) and (4.3) is generally incompatible. Our problem consists in finding the value of ω for which it becomes compatible. From the first five equations of these systems, we find, respectively, that

$$x_1 = 0{,}482,\ x_2 = 0{,}775,\ x_3 = 1{,}044,\ x_4 = 1{,}160.\ x_5 = 0{,}867$$

and that

$$x_1 = 0{,}529,\ x_2 \quad 0{,}938,\ x_3 = 1{,}069,\ x_4 = 0{,}784,\ x_5 = -0{,}023.$$

The last equation of this set takes on the values $\Delta = -0.879$ and $\Delta = 2.729$, respectively in the case of the given values of the variables $x_1, x_2, \ldots, x_5$. Carrying out a similar calculation

for $\omega = 9.9$, we find: $\Delta = -0.224$. Therefore the desired solution ω is equal to 9.9. For $\omega = 9.9$, $x_1 = 0.485$, $x_2 = 0.787$, $x_3 = 1.013$, $x_4 = 1.053$, $x_5 = 0.680$ and one can consider that the assumption made regarding the values of $x_1, x_2, \ldots, x_5$ is satisfied.*

*We note that the calculation of this example by the method of harmonic balance gives a value equal to approximately 1.3 for the amplitude.

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EXPERIMENTAL DETERMINATION OF THE EQUIVALENT COMPLEX TRANSMISSION COEFFICIENTS OF NON-LINEAR SECTIONS OF CONTROL SYSTEMS

I. N. Pechorina

(Abstract) A description is given of the method and circuit arrangements for determination of the equivalent complex transmission coefficients of non-linear sections of control systems. It is advantageous to use this method when a sinusoidal oscillation is appreciably distorted in passage through a non-linear section.

A method is described in [1] for the determination of the equivalent complex transmission coefficients of non-linear quadripoles. In certain cases, this method can be used also for the analysis of non-linear sections of control systems [2]. However, the necessity of the selection of the transmission function of the linear section with such a calculation in order that auto-oscillations can

~~arrive~~ take place in the given system makes difficult the use of the detailed method for the investigation of complicated non-linear sections of control systems.

In the present paper a method is considered for the experimental determination of the equivalent complex transmission coefficients of non-linear sections with the help of Lissajou figures. The method outlined can be used for obtaining a family of frequency characteristics in the case when the output oscillation differs from sinusoidal. Experimental determination of transmission coefficients makes it possible to bring accuracy to the expressions for these coefficients obtained by computational methods.

1. For the determination of the equivalent complex transmission coefficients, it is necessary to have a low frequency generator which simultaneously yields two oscillations which differ in phase by 90°. Electronic low frequency oscillators which are widely used for obtaining the frequency characteristics of systems make it possible to obtain sinusoidal and co-sinusoidal oscillations simultaneously. In the use of a selsyn with drive from a constant current motor with a wide range of regulation of the velocity as the source of low frequency oscillations, it is necessary, to obtain component voltages differing in phase by 90°, to introduce a small addition into the circuit.

The determination of the complex transmission coefficients is carried out in the following way (see Fig. 1). A sinusoidal

voltage u_1 is applied to the input of the given section 1. At the output of this section a periodic osciliation $f(\omega t)$ is obtained which differs from sinusoidal. The oscillation under investigation is applied to the vertical plates of the oscilloscope 2, the tube of which has a long persistant screen. Scanning is carried out for the voltage u_1 (position 1 of the switch K_2) or the voltage u_2 (position 2 of the switch K_2). The voltages u_1 and u_2 are harmonic and differ in phase by 90° relative to one another. The switch K_1 is used for calibration; with it one can key in upon the vertical input the voltage $f(\omega t)$ (position 1) or u_1 (position 2).

A possible Lissajou figure is shown in Fig. 2, obtained for a sinusoidal sweep voltage $f(\omega t)$. The area of this closed figure S_1 is determined in the following fashion:

$$S_1 = \int_0^{2\pi} f(\omega t)\ b \cos(\omega t)\ d(\omega t). \tag{1}$$

In an entirely similar way, one can obtain the area of the Lissajou figure for a cosinusoidal sweep voltage $f(\omega t)$:

$$S_2 = \int_0^{2\pi} f(\omega t)\ b \sin(\omega t)\ d(\omega t). \tag{2}$$

In equations (1) and (2), b is the amplitude of the oscillation with whose help the sweep is obtained.

We expand the oscillation $f(\omega t)$ in a series:

$$f(\omega t) = f_0 + c_1 \sin \omega t + c_2 \cos \omega t + \ldots$$

The coefficients c_1 and c_2 are determined by the equations

$$c_1 = \frac{1}{\pi} \int_0^{2\pi} f(\omega t) \sin(\omega t)\, d(\omega t);$$
$$c_2 = \frac{1}{\pi} \int_0^{2\pi} f(\omega t) \cos(\omega t)\, d(\omega t). \quad (3)$$

Comparing Eqs. (1), (2), and (3), we see that

$$c_1 = S_2/\pi b;$$
$$c_2 = S_1/\pi b. \quad (4)$$

From Eqs. (4), we obtain the imaginary and real parts of the complex transmission coefficient W_H of the non-linear section under study for an amplitude of the input oscillation equal to a:

$$\mathrm{Re}\ W_H = S_2/\pi ab;$$
$$\mathrm{Im}\ W_H = S_1/\pi ab. \quad (5)$$

2. The circuit for the determination of the equivalent complex transmission coefficients is shown in Fig. 3. By means of the reduction 2, the constant current motor 1 operates the selsyn 3. The modulated voltage taken by the selsyn is applied to the input of the phase-sensitive amplifier 4. The voltage at the output of the amplifier is smoothed by the filter 5 and can be applied to the input of the linear section 6 whose characteristics are determined. Parallel to the basic selsyn, the same type of selsyn 3 is keyed in (with amplifier 4' and filter 5). The voltage, taken with the single phase winding of the selsyn 3', is transformed in the same way as the voltage of the fundamental selsyn. In this way two identical

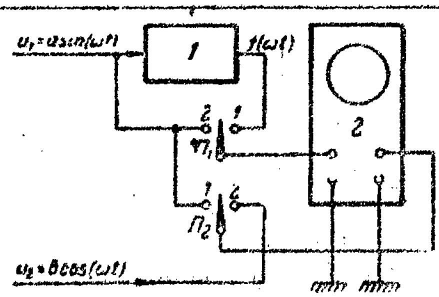

Fig. 1.

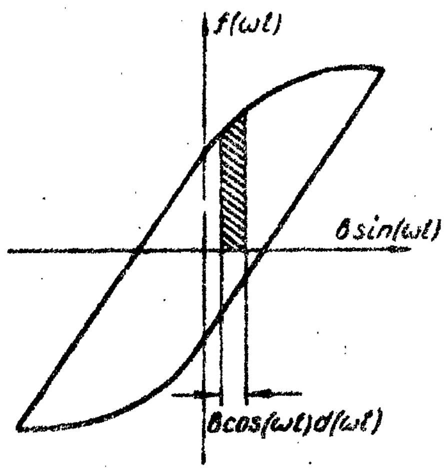

Fig. 2.

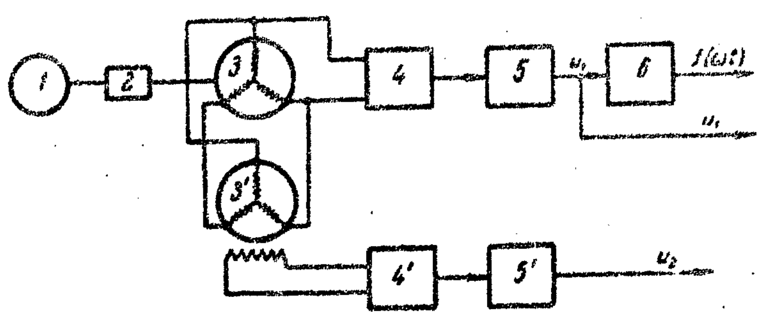

Fig. 3.

channels are obtained. By rotating the rotor of the second selsyn, it is possible at the output of the second channel to establish a voltage of low frequency shifted by 90^{o} relative to the voltage at the output of the first channel. For the second channel one can successfully use test apparatus of the selsyn.

The shift in phase of the voltages of low frequency at the output of the first and second selsyns is somewhat changed upon increase in the frequency, which requires either a special calibration of the rotation of the rotor of the auxiliary selsyn or confirmation of the correctness of the phase of the voltage u_2 relative to the voltage u_1 by measurement at a given frequency.

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HIGH DENSITY RECORDING OF NUMERICAL INFORMATION ON A MAGNETIC DRUM

(Abstract) The results are given for the investigation of the recording of numerical information on a magnetic drum with density of 16-20 entries per millimeter.

The density of recording on a magnetic drum depends on the construction of the magnetic head, on the size of the gap between the tape and the magnetic head, the travel rate of the tape relative to the head, the length of the signals of the recording, the quality and thickness of the magnetic tape. Most of the factors enumerated are interrelated. At the present time the chief factors limiting increase in the density of the recording on the magnetic drum are the construction of the magnetic head and the size of the gap between the head and the covering.

The resolving power of the magnetic head is determined by

the dimensions of the regions in which the magnetic field H, which borders the gap of the head in its excitation by a small current, is important (for example, $H \geq 0.05$ oersteds, see Appendix). To increase the resolving power of heads the latter must have a construction which guarantees a sharp fall-off in the fringe field in a small region. For obtaining a recording density of 10 signals per millimeter, it is necessary that the field of the head be localized in a region of width less than 100 microns. One of the ways of decreasing the fringe field consists in the decrease in the thickness of the pole pieces and the forward gap of the head. However, it is necessary to take into account that, in a small gap, its magnetic reluctance and the reluctance of the magnetic circuit of the head become comparable in value. In reproduction brought about by the tape, the magnetic flux is split up: part of the flux closes the circuit through the forward gap of the head, escaping the magnetic circuit. It is natural that the effectiveness of the head in reproduction is the lower, the larger the part of the flux which escapes through the gap.

To obtain a satisfactory print on the magnetic tape upon reduction of dimensions of the fringing field, it is necessary to reduce to a minimum the gap between the tape and the head. It is difficult to reduce the eccentricity in the manufacture of the drum to less than 3-4 microns. To guarantee a stable gap of the size of several microns under these conditions is possible only on preparation of a

block of heads which automatically follow the surface of the drum [5].

1. MAGNETIC HEAD

The construction of the magnetic head connected to a transformer is shown in Fig. 1. The magnetic head is a "horseshoe" of a material with high magnetic permeability (permalloy etc). The winding of the head consists of a single coil and is simultaneously the secondary winding of the transformer. The structural joining of the magnetic head and the transformer was done to avoid losses in the transmission of large recording currents. The ratio h/a = 5-25 varying with the thickness of the material of the magnetic circuit. The core of the transformer is made out of ferrocart of 3 mm diameter (the magnetic permeability $\mu = 1000$). The primary winding of the transformer consists of 45 turns of wire PEV of 0.1 mm. The secondary winding of the transformer is a square winding of copper foil of 15 microns thickness. The lower part of this winding is wrapped around by a "horseshoe" of permalloy of thickness 10-20 microns and is clamped in a chuck. The width of the pole tip of the head is 1 mm. The inductance of such a head in assembled form amounts to about 9 microhenries.

The heads were tested on iron-lacquer and nickel-cobalt coatings. In the use of the iron-lacquer coating, the signal-to-noise ratio was somewhat poorer than for the nickel-cobalt. The linear speed of the tape was $\sim 30 m \cdot sec^{-1}$. The recording circuit made it possible to deliver a bundle of pulses synchronized with a marker

pulse previously recorded on another track. The period of repetition of pulses in the packet varied from 1.5 to 20 microseconds; the length of the pulse of the recording current varied from 0.4 to 1.5 microseconds. The amplitude of the recording current varied from 0 to 4a.

For pole tips of the magnetic head made out of non-annealed permalloy 50 NKhS of 20 microns thickness and with a gap between the tape and the head of less than 5 microns (concerning the gap, see below) the recording density of digital information of 10 units per mm was obtained without distortion of the amplitude. Oscillograms of a single pulse and the code 1111011111 are shown in Figs. 2a and 2b (the repetition frequency of the pulses was 300 kc, the length of the current recording was 0.4 microseconds, the amplitude 3a; the output signal had an amplitude of 1 mv). Figures 2c and 2d show oscillograms of signals for a thickness of the pole points of the head of 10 microns. The amplitude and the length of the recording current were the same as in the previous case. The recording density was 16 pulses per millimeter. Figure 2e shows the oscillogram of signals obtained with the same head for a recording density of 20 signals per millimeter. It is seen from the oscillogram that a distortion of the signal takes place. Making use of differentiation in the formation of the signals under discussion [1], one can obtain a completely reliable division of zeros and units even for frequent superposition of signals. Figure 2f shows an oscillogram

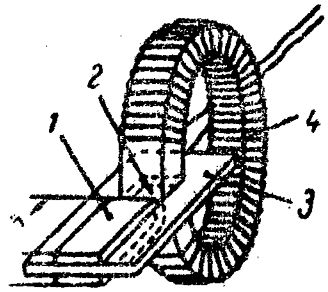

Fig. 1.

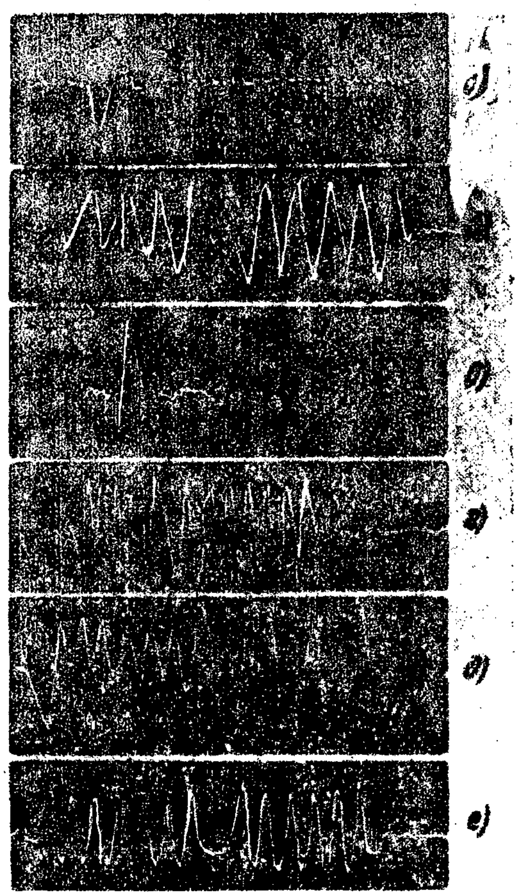

Fig. 2.

of the signals formed for a density of 20 items per millimeter.

The comparatively small amplitude of the signal from the head is not a serious drawback. The head possesses a small impedance, ~~and~~ which makes it possible to use a step-up transformer which is applied in the recording and in the reproduction.

The head operates with high current densities. However, the yoke of the head is intensively abraded by air which is entrained by the surface of the drum and is excellently cooled.

For the determination of the optimal amplitude of recording current, the dependence of the considered signal on the amplitude of the recording current was removed. The comparatively large recording current (about 3a) can be explained by the fact that a magnetizing force much larger than the coercive force is necessary to bring the system magnetic head-tape into a definite magnetic state after a small interval of time.

2. AUTOMATIC MAINTENANCE OF THE GAP

Reduction of the limiting gap of magnetic heads requires reduction in the gap between the head and the tape. Maintenance of a constant gap at a magnitude of several microns is a difficult engineering problem. Application of thermal compensation and very rigid requirements applied to the construction of the drum do not give sufficiently satisfactory results. In this connection, conversion to a floating suspension of the head is of interest.

If an infinite plane moves relative to an inclined plane in a

viscous incompressible medium, then a force arises which deflects the plate away from the plane [2]. For a given inclination of the plate, the resulting pressure is inversely proportional to the square of the size of the gap between the plate and the moving plane. If the plate is loaded by a force P, then some equilibrium gap is established for which a balancing of the external force and the internal force due to the fiscosity of the medium takes place. If the gap is changed for any reason, then an extra force appears tending to restore the plate to the equilibrium position. In the case of a rotating cylinder and a plate, the process which takes place is similar.

The model on which the work was carried out consisted of two drums of diameters 600 mm and 200 mm with a floating suspension of blocks of magnetic heads. The suspension of the floating blocks consisted of a fork, frame and the floating block itself. This system formed a "universal joint," which made it possible for the block to be freely moved parallel to the generatrices and to follow after the surface of the drum. The suspension was made as light as possible in order that its inertia did not affect the stability of the gap in the presence of the eccentricity of the drum. A system is employed in the apparatus which makes it possible to clamp the floating block to the surface of the drum with a given force. Along with the suspension, this system is in aerodynamic balance.

Attempts to measure the value of the gap by means of a beam of light directed through it did not lead to positive results

since in the most interesting region of gap values (~ 5 microns) the quantity of light passing through the gap is very small in comparison with the scattered light reflected from the details of construction. Therefore, estimate of the value of the gap was performed by an indirect method. The value and the stability of the gap ~~was~~ were estimated from the amplitude of reproducible signals (recording and reading were carried out by the same head). From the viewpoint of magnetic recording, the reproducible system is the final "product," since in the given case a similar approach is justified.

Tests of the floating block of magnetic heads were carried out on a drum of diameter 600 millimeters (iron-lacquer covering) and a drum of diameter 200 millimeters (nickel-cobalt covering). The linear velocity of the surface of the drum was 30 m/sec. The area of the side of the block turned toward the magnetic drum was 10 X 15 mm.

The maximum pressing force P_{max} for which the block of heads "lies down" on the drum depends strongly on the shape of the surface of the block turned toward the drum. In the case of a plane surface, for a drum of diameter 600 mm, P_{max} is equal to 190 g, for a drum of diameter 200 mm, it is equal to 50 g. Decrease in the pressing force is explained by the increased curvature of the surface of the drum. The comparatively large curvature of the drum leads to the result that the phenomenon is strongly distorted

in comparison with a plate moving relative to the plane.

The maximum hydrodynamic force arises for a very definite ratio of the initial and final cross sections; therefore, it would be profitable to make the surface of the drum and the internal surface of the block congruent. Experiments confirm this suggestion: for floating blocks of the same profile as in the first case, ground to the contact surface of the drum, the maximum pressing force was increased to 500 and 450 g, respectively.

Oscillograms are shown in Fig. 3 for a series of pulses recorded and reproduced by a head rigidly fixed relative to the surface of the drum and by a similar head which automatically followed the surface of the drum. As is evident from the oscillogram, the floating suspension of the block of heads entirely eliminates the modulation of signals produced by the beat of the drum in the case of rigidly heads.

The results we have set forth show that the construction of a block of heads which automatically follows the surface of the drum in conjunction with slightly shaped magnetic heads makes it possible to increase the recording density of numerical information on a magnetic drum up to 16-20 items per millimeter.

APPENDIX

ESTIMATE OF THE RESOLVING POWER OF A MAGNETIC HEAD

For two circuits linked by a common magnetic flux, it follows from the reciprocity theorem that

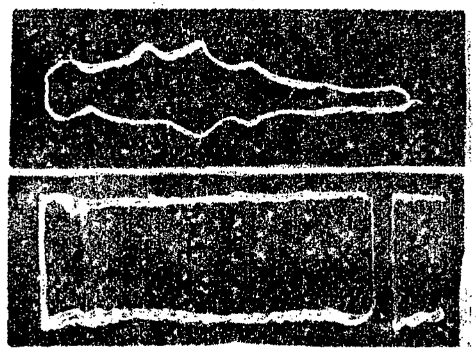

Fig. 3

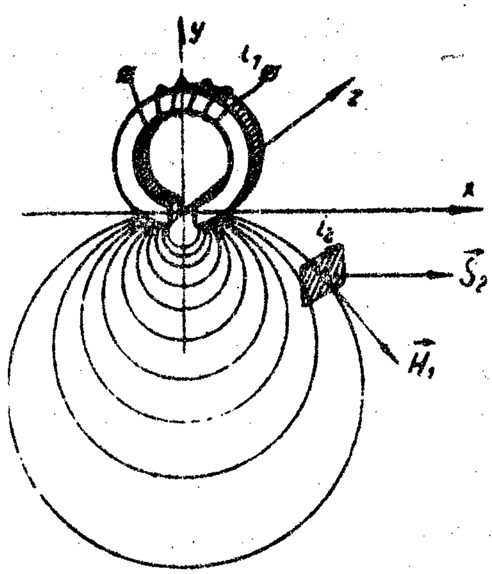

Fig. 4.

$$\Phi_{1,2} = M_{1,2}\, i_2; \quad \Phi_{2,1} = M_{2,1}\, i_1, \tag{1}$$

where $\Phi_{1,2}$ is the flux through the surface enclosed by the first circuit and produced by the current i_2 in the second circuit, and $\Phi_{2,1}$ is the flux through the second circuit created by the current i_1 in the first circuit, $M_{1,2} = M_{2,1}$ is the coefficient of mutual inductance.

Let one circuit be the winding of the magnetic head and the second circuit be a loop in the transverse cross section of the magnetic tape (see Fig. 4). We assume that

1) there is a current in the circuit of the head which is sufficiently small that the problem can be assumed linear (this current produces the magnetic field $\underset{\sim}{H}_1(x, y, z)$ in space);

2) the magnitude of this current is equal to unity;

3) the cross section S_2 of circuit 2 is sufficiently small that $\underset{\sim}{H}_1$ can be considered to be constant over the cross section in this case;

4) the problem is two-dimensional ($\partial H/\partial z = 0$).

For unit current in the first circuit, the flux $\Phi_{2,1}$ through the second circuit will be equal to $\Phi_{2,1} = \underset{\sim}{S}_2 \cdot \underset{\sim}{H}_1$; similarly, for any current in the first circuit,

$$\Phi_{2,1} = S_2 H_1 i_1. \tag{2}$$

We write down the components of the flux along the coordinate axes

$$\Phi_{2,1x} = S_{2x} H_{1x}\ (x,\ y)\ i_1;$$
$$\Phi_{2,1y} = S_{2y} H_{1y}\ (x,\ y)\ i_1.$$

If we compare Eqs. (1) and (2), it is easy to see that

$$M_{2,1x} = S_{2x}H_{1x}\ (x,\ y);$$
$$M_{2,1y} = S_{2y}H_{1y}\ (x,\ y).$$

If we take it into account that $M_{2,1} = M_{1,2}$, we can solve the inverse problem, i.e., we can compute the flux through circuit 1 from a current in circuit 2:

$$\Phi_{1,2} = \mathbf{S}_2\mathbf{H}_1\ (x,\ y)\ i_2.$$

It is known that $\mu \underset{\sim}{S}_2 i_2 = \underset{\sim}{m}_2$, where $\underset{\sim}{m}_2$ is the magnetic moment of circuit 2. Therefore,

$$\Phi_{1,2} \backsim \mathbf{m}_2\mathbf{H}_1\ (x,\ y) \tag{3}$$

or, in the components along the coordinate axes,

$$\Phi_{1,2x} \backsim m_{2x}H_{1x}\ (x,\ y);$$
$$\Phi_{1,2y} \backsim m_{2y}H_{1y}\ (x,\ y).$$

From the relations we have written down, it is seen that $\underset{\sim}{H}_1(x, y)$ (the intensity of the field created in the first circuit by the passage through it of unit current) emerges in the given case as a measure of the magnetic coupling between the source of the magnetic flux (circuit 2) and the magnetic head (circuit 1).

Let us consider a specific magnetic print, that is, a certain distribution of the magnetization $\underset{\sim}{J}$ on the magnetic tape. Let $\underset{\sim}{J}$ not vary with the depth of the tape and the transverse path: the thickness of the tape and the width of the track are constant. We limit ourselves to a consideration of the linear case as the most typical, i.e., we shall assume that $\underset{\sim}{J} = f(x)$.

We select a layer normal to the x axis of thickness dx so that we can consider J(x) = const. in this region.

Taking it into account that the magnetization--the volume density of the magnetic moment--can be written for small volume of the layer under consideration as

$$\sum m = J dV \backsim J dx,$$

where dV is the volume of the layer in question, $\sum m$ is the sum of the elementary magnetic moments contained in the considered volume. In view of the smallness of the volume, the vector m can be considered as the magnetic moment of the layer. Then, taking (3) into account and assuming that the magnetic printing is displaced relative to the coordinate origin by the quantity $\bar{x}$, we conclude that the flux produced by all of the printing is

$$\Phi_{1,2x} \backsim \int_{-\infty}^{+\infty} J_x\,(x-\bar{x})\,H_{1x}(x, y)\,dx;$$

$$\Phi_{1,2y} \backsim \int_{-\infty}^{+\infty} J_y\,(x-\bar{x})\,H_{1y}(x, y)\,dx. \qquad (4)$$

The limits of integration in (4) can be contracted to the region in which the integrand differs from zero.

Let the magnetic printing move past the head with velocity v. This produces a change in the interlinkage in the head and to the appearance at the end of the coil of an emf

$$e_x(t) = \frac{d\Phi_{1.2x}(t)}{dt} N,$$

where N is the number of turns in the coil. Noting that $\bar{x} = vt$, we obtain:

$$e_x(\bar{x}) = \frac{d\Phi_{1.2x}}{d\bar{x}} Nv.$$

Substituting the value of $\bar{\Phi}_{1,2x}$ from (4), we have:

$$e_x(\bar{x}) = Nv \frac{d}{d\bar{x}} \int_{-\infty}^{+\infty} H_x(x, y) J_x(x-\bar{x}) dx; \tag{5}$$

similarly,

$$e_y(\bar{x}) = Nv \frac{d}{d\bar{x}} \int_{-\infty}^{+\infty} H_y(x, y) J_y(x-\bar{x}) dx.$$

Let us consider a specific case of a step-wise change in the magnetization along the tape:

$$J_x(x) = J_0 1(x).$$

This case is quite characteristic for recording of numerical information.

Substituting the latter expression in (5), we obtain:

$$e_x(\bar{x}) \backsim J_{0x} Nv \frac{d}{d\bar{x}} \int_{-\infty}^{+\infty} H_x(x, y) 1(x-\bar{x}) dx =$$

$$= J_{0x} Hv \int_{\infty}^{+\infty} H_x(x, y) \delta(x-\bar{x}) dx.$$

The derivative of the step function is the delta function of Dirac:

$$\delta\,(x-\bar{x})=1 \text{ for } x=\bar{x};$$

$$\delta\,(x-\bar{x})=0 \text{ for } x\neq\bar{x}.$$

Consequently,

$$\int_{\infty}^{+\infty}\delta\,(x-\bar{x})\,H_x(x,\,y)\,dx=H_x\,(\bar{x},\,\bar{y});$$

$$e_x\,(\bar{x})\backsim J_{0x}\,Nv H_x\,(x,\,\bar{y}),$$

that is,

$$e_x\,(x)\backsim H_x\,(x,\,y).$$

Similarly,

$$e_y\,(x)\backsim J_{0y}\,NvH_y\,(x,\,y),$$

that is,

$$e_y\,(\bar{x})\backsim H_y\,(\bar{x},\,\bar{y}).$$

Thus if the printing on the tape has the form of a jump and if the magnetization is oriented strictly along the tape, then the signal repeats in its form the dependence of $H_x(x, y)$. The width of the reproduced impression $l = \Delta tv$ in this case depends only on the configuration of the field bordering the gap of the magnetic head. (By a wide impression l is understood a region inside which $|H_x| \geq \epsilon$.) These considerations are illustrated in Fig. 5.

It is clear from the foregoing that the edge field must be localized in the narrowest possible range. Actually, the magnetization has a component along the x axis and along the y axis which is the same as the real field of the head:

$$H_x = |\underset{\sim}{H}|\cos\alpha; \quad H_y = |\underset{\sim}{H}|\sin\alpha,$$

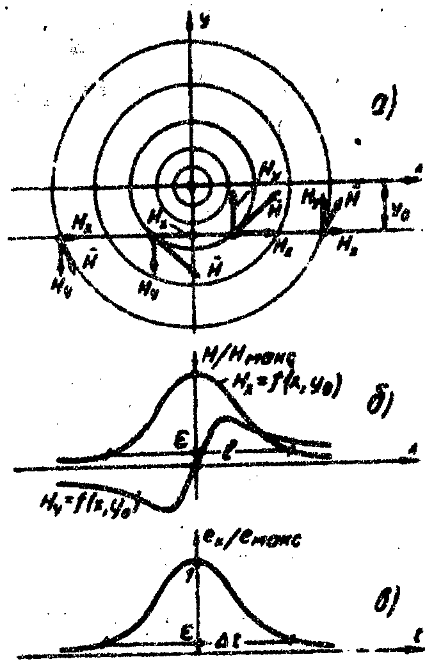

Fig. 5.

where α is the angle between the direction of the vector $\underset{\sim}{H}$ and the x axis.

The resulting signal e from the magnetic head will be equal to the algebraic sum of the two components $e = e_x + e_y$. With increased distance from the axis of symmetry of the head along the x direction, the field component H_x falls off rapidly since $|\underset{\sim}{H}|$ and $\cos \alpha$ both fall off; the component H_y falls off more slowly since $\sin \alpha$ increases in this case (see Fig. 5).

For improvement of the resolving power of the magnetic head for reading, it would be expediant to make use only of the field component H_x. There are similar recommendations in the literature [3, 4]; however, practical use of only one field component is extremely difficult. Therefore, localization of both components of the field of the head in the shortest possible interval would be a substantial contribution.

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ON THE STRUCTURE OF ELECTRONIC INHOMOGENEITIES IN THE OUTER CORONA OF THE SUN

(This is a translation of an article written by V. V. Vitkevich, B. N. Panovkin and A. G. Sukhoveĭ in Radiofizika (Radiophysics), Vol. II, No. 6, 1959, pages 1005-1007.)

(Submitted to editor December 7, 1959)

1. Studies of the outer corona of the sun by the method of penetration of it by radio waves have to date been carried out chiefly by means of radio interferometers located approximately in the east-west direction. Such observations have made it possible to measure the size of the effect of scattering in a single direction only. On the basis of these observations, data have been obtained on the electronic concentration of the inhomogeneities of the outer corona, and the dependence of the dimension of the inhomogeneities of the corona on the phase of the eleven-year cycle of the sun's activity has also been discovered [1-3].

However, it must be noted that for investigation of the form of the scattered source, it is necessary to determine its dimensions in different directions, i.e., to use radio interferometers with base

lines of different orientations. The desirability of such investigations has been shown in reference [1]. Observations of such a nature (not completely successful because of interference) were carried out in 1954 at the Crimean station of the Physics Institute of the Academy of Sciences, while more reliable results were obtained with an interferometer oriented in the north-south direction [5].

However, comparatively little data were obtained from these observations on the form of the scattered source, although the results led to the conclusion that the effect of scattering is more marked in the direction perpendicular to the radius vector which connects the source of the radio emission and the sun. The conclusion was then drawn on the extension of the inhomogeneities in the radial direction from the sun [4, 6].

In 1958, the above-mentioned observations were carried out by us at two wavelengths (λ = 3.5m and λ = 5.8m).

2. At the Crimean station of the Physics Institute of the Academy of Sciences, observations after covering of the source Taurus-A of the outer corona were carried out at a wavelength of 5.8m with two radio interferometers; one of these, described previously [3] was pointed in the east-west direction (the observations were made by M. A. Ovsyankin), while the other interferometer was directed at an angle to the first such that the direction of the base line made an angle of 14.5° with respect to the north-south direction. The length of the base line was 863 m. The antenna

were connected by a high frequency cable. For compensation of the damping accompanying the cable, amplifying units were located under each of the antenna. The receiving apparatus operated by a modulation method. The band width of the receiver was $\Delta f = 0.5$Mc., the sensitivity $\Delta T \sim 2^{\circ}K$ and a time constant of about 12 seconds. An automatic electronic potentiometer was used as a recording instrument. The speed of motion of the chart paper was 36cm/hour. The feed of the receiving system was stabilized at the anode and at the cathode. Control of the exactness and stability of the amplification factor of the apparatus was established daily by reception of the radio emission of a source in the constellation Virgo and also by means of noise generators.

Systematic observations were carried out during almost the whole of June, 1958. During the periods of observation, the radio emission of the sun was quiet and did not interfere with the reception of the radio emission from a source in the constellation Taurus. Only interference bothered the observations, as a result of which less data were obtained for the second phase of covering. The curve of the intensity of radio emission for the source in the constellation Taurus, which was covered by the sun's corona for the first and second radio interferometers is given in Fig. 1. As is seen from the figure (see curve 1), a decrease of intensity on the second interferometer was noted as early as June 8; thereafter the intensity fell off and reached a minimum value on 14-16 June;

then the intensity increased and from 24 June the intensity became stable. However, observations of this year on the base line east-west (curve 2, Fig. 1), and also observations of previous years on the horizontal base lines have shown that a decrease of intensity of the source is usually observed from 11 or 12 June (in 1957, from 10 June). Thus observations with a diagonal base line gave entirely reliable results consisting of the fact that the effect of scattering of radio waves is observed much earlier than on the east-west base lines. If we project the direction of the base line on the celestial sphere (Fig. 2), then it is easily shown that in the first phase of an eclipse we observe scattering on the base line in a direction close to the perpendicular relative to the line connecting the source and the sun.* It therefore follows that the scattering effect is more clearly expressed in a direction close to the perpendicular (Fig. 2a) than in a direction close to the horizontal (Fig. 2b); i.e., the inhomogeneity of the outer corona has approximately a radial structure.

*In the second phase of the eclipse the scattering angles are also shown to be large for the second base line. However, these data require more exact treatment and detailed confirmation in view of the presence of interference.

3. At the Serpukovski radio-physical station of the Physics

Institute of the Academy of Sciences the same apparatus was used for observations at a wavelength of $\lambda = 3.5$m as in 1957. The observations were carried out morning and evening by an interference method with a base line of 320 m (i.e., 95λ). The method of observation was shown in [4]. This time we succeeded in obtaining more reliable results at a wavelength of 3.5m than during the previous year. Unfortunately, strong interferences did not allow us to carry out the evening observations satisfactorily.

The materials obtained made it possible to construct a curve of the change in the relative percentage modulation from the morning observations (see Fig. 3). The evening observations gave several points whose position differed from the curve of the variation of the morning observations.

It follows from Fig. 3 that

a) the beginning of a significant change of the relative percentage modulation was noted as early as 8 June;

b) some lack of symmetry exists for the curve relative to the maximum of covering (this observation is quite trustworthy);

c) in the first phase of the eclipse the morning observations gave a relatively higher percentage modulation than the evening. This result is interesting to compare with the position of the projection of the base line of the interferometer on the terrestrial sphere during the time of observations which is done in Fig. 4.

The measurements given at wavelengths of 3.5m and 5.8m

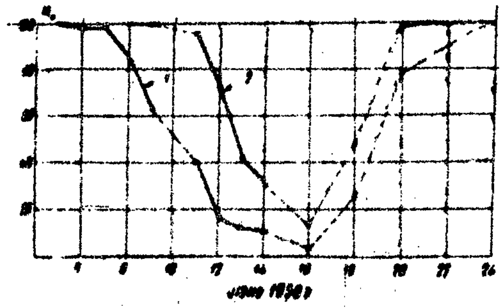

Fig. 1. Received intensity of radio emission of the source *Taurus*-*A* (λ = 5.8m):

1--base line in the east-west direction; 2--base line in the north-south direction

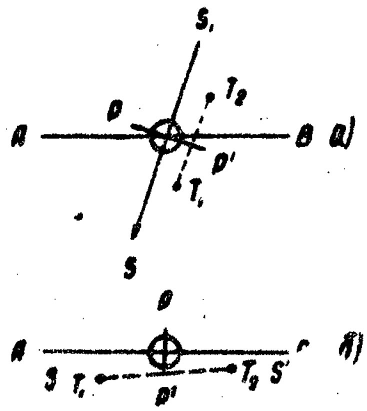

Fig. 2. Projections of the base lines of the radio interferometer on the celestial sphere (λ = 5.8m):

a) the base line is oriented in the north-south direction; b) the base line is oriented in the east-west direction (S is the sun, T_1 and T_2 are the positions of the source *Taurus*-*A* before and after the eclipse. AB is the projection of the base line of the interferometer; ss' are the daily motions of the sun, p, p' the band of the sun.

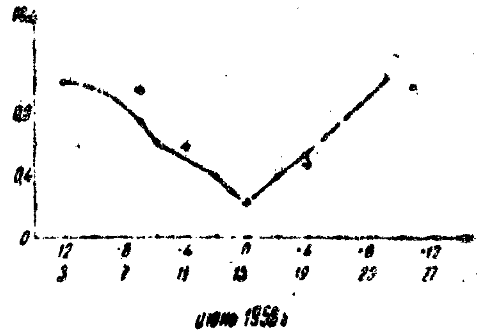

Fig. 3. Received intensity of radio emission of the source

($\lambda = 3.5$m):

.... - morning observations, +++- evening observations.

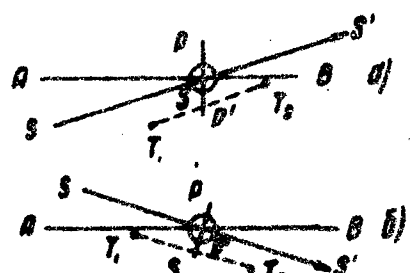

Fig. 4. Projection of the base line of the interferometer on the celestial sphere ($\lambda = 3.5$m). The notation is the same as in Fig. 2.

undoubtedly support the fact that the radiation is non-isotropic. The electronic inhomogeneities by which the scattering is produced have an extended form and are located chiefly in a radial direction relative to the sun.

At the present time more detailed treatment of the data is being carried out.

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DETERMINING THE DENSITY OF THE NEUTRAL COMPONENT IN THE IONOSPHERE

(This is a translation of an article written by V. G. Kurt in Radiofizika (Radiophysics), Vol. II, No. 6, 1959, pages 1007-1009.)

Recently, our representations of the basic physical characteristics of the upper atmosphere have undergone considerable changes. Data on the density distribution of the neutral component have suggested the presence of a density approximately ten times higher than had been assumed two years earlier. These results were obtained primarily by means of artificial earth satellites [1, 2].

However, the determination of the density of both ionized and neutral components is necessary by other independent methods for experimental proof. It is desirable that such a method allow one to obtain the fundamental characteristics of the earth's atmosphere in a single measurement, i.e., its density and temperature. Such an experiment was suggested by I. S. Shklovskii. The idea of the experiment is an extraordinarily simple one. There is an ejection of sodium vapors from high altitude rockets at the altitude of interest to us in the morning or evening twilights.*

*A similar method was applied earlier in the USA for the determination of wind velocity and other parameters of the atmosphere at altitudes of the order of 100 km. The problem of the determination of the density of the atmosphere was not set forth in these experiments.

A cloud, radiated by the sun, shines as a result of resonance fluorescence, which makes it possible to observe it from the earth's surface. In our case, the ejection of the vapors was carried out at an altitude of 430 km above the earth. The altitude of the earth's shadow during the time of the experiment was 300 km. Observations were made over an interval of 15 minutes in which there were obtained about 50 photographs of the cloud. After this time, the cloud reached the dimensions of several hundred kilometers. Since each atom of sodium undergoes a number of collisions with the atmospheric atoms, then within the short interval of time after the ejection of the sodium vapor (it can be shown that this time is equal to 100 seconds) a diffusion distribution is established in the cloud.

However, it must be kept in mind that the diffusion of sodium in the atmosphere takes place in an inhomogeneous medium in the presence of gravitational forces. In this case, the diffusion equation is written in the form

$$\frac{\partial u}{\partial t} = \operatorname{div}(D \operatorname{grad} u) + \operatorname{div}(V_z u) - C, \qquad (1)$$

where u is the concentration of sodium atoms, D is the diffusion coefficient, V_z is the drift velocity of the sodium atoms in the gravitational field, C is the difference between the number of sodium atoms ionized by the ultraviolet radiation of the sun and the number which become atoms again. In our case, C can be set equal to zero since the lifetime of the sodium atoms in the field of the sun's radiation is equal to several hours.

Equation (1) was solved by us for two different initial conditions: 1) diffusion begins at the moment t = 0 when the distribution of particles has the form

$$u = Q\delta(z)\frac{\delta(r)}{r},$$

where Q is the total number of atoms of sodium in the cloud; 2) diffusion begins at the moment $t = t_0$ when the sodium atoms fill uniformly a sphere of radius R_0.

To begin with, we simplify the equation by setting $V_z = 0$, D = const. Then the solution in the first case has the very simple form:

$$u = \frac{Q}{4\sqrt{\pi}(Dt)^{3/2}} e^{-R^2/4Dt}. \qquad (2)$$

Comparing Eq. (2) with the dependence of u on R found from experiment, one can obtain the value of the diffusion coefficient D. The latter is connected in turn with the density of the atmosphere n:

$$D = \frac{3\pi}{32} \frac{1}{nQ_d} \sqrt{\frac{8kT}{\pi}\left(\frac{M_1 + M_2}{M_1 M_2}\right)}. \tag{3}$$

where Q_d is the effective diffusion coefficient, and T is the temperature of the atmospheric atoms, M_1, M_2 are the masses of the atoms of the atmosphere and of sodium.

Substituting numerical values for the altitude of 430 km in (3), and assuming that the atmosphere at this altitude consists of atomic oxygen (of which we have complete proof), we find for the concentration of atmospheric atoms the value 0.8×10^8 cm^{-3} as a first approximation.

Furthermore, we shall improve this result by giving up the exact initial condition. In this case the solution for our centrally symmetric problem will be*

$$u(0, t) = u(0, 0)\left[\psi\left(\frac{R_0}{\sqrt{2Dt}}\right) - \frac{R_0}{\sqrt{4\pi Dt}} e^{-\frac{R_0^2}{4Dt}}\right]. \tag{4}$$

*Here $\psi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ is the probability integral.

Comparing the concentration in the center of the cloud found by experiment with the expression (4), we find that the best agreement is obtained for $n = 1.8 \times 10^{-8}$ cm^{-3}. Detailed description of the method of finding the concentration of sodium atoms is given in [3].

We shall now estimate the effect of the force of gravity and the density gradient of the atmosphere. For simplicity, we shall limit ourselves to the one-dimensional case with exact initial conditions. Equation (1) will have the solution

$$u(z, t) = \frac{Q}{2\sqrt{\pi D_0 t}}\, e^{-(z - V_z t)^2/4D_0 t} \qquad (5)$$

for the case of a homogeneous atmosphere in the presence of gravitational forces, i.e., for $D = D_0$ = const, and V_z = const $\neq 0$. In (5), $V_z = \lambda g/v_t$, λ is the mean free path length of the particle, g is the acceleration due to gravity, v_t is the thermal velocity. For the case of an inhomogeneous atmosphere with a constant temperature and without account of gravitational forces, i.e., for $D = D_0 e^{\alpha z}$, $V_z = 0$, the solution of (1) has the form:

$$u = \frac{\alpha Q}{D_0 \alpha^2 t} \exp - \left(\frac{\alpha z}{2} + \frac{1 + e^{-\alpha z}}{D_0 \alpha^2 t} \right) I_1 \left(\frac{2e^{-\frac{\alpha z}{2}}}{D_0 \alpha^2 t} \right). \qquad (6)$$

where α is a quantity which is the inverse height of a homogeneous atmosphere, D_0 is the diffusion coefficient for the altitude at which ejection of the sodium vapors takes place, I_1 is the Bessel function of imaginary argument. In the first case, we shall have a uniform descent of the cloud to below with a velocity V_z which for our experiment is equal to 3×10^3 cm/sec. The form of the cloud in this case is preserved just as in the absence of gravitational forces.

In the second case, the cloud is deformed while the region of maximum concentration is shifted downward at a decreasing

velocity. The position of the point of maximum concentration is found from the equation:

$$I_1\left(\frac{2e^{-\frac{\alpha z}{2}}}{D_0\,\alpha^2 t}\right)e^{\frac{\alpha z}{2}} = I_0\left(\frac{2e^{-\frac{\alpha z}{2}}}{D_0\,\alpha^2 t}\right), \qquad (7)$$

where I_1 and I_0 are Bessel functions of imaginary argument.

Figure 1 shows the dependence of the value of the descent of the cloud on the time in non-dimensional coordinates.

The effect of gravitational forces and the presence of the density gradient of the atmosphere lead to a shift in the effective center of the sodium cloud by less than 50 km after 1000 sec, which can be completely neglected, especially if the observations last for about 500 sec.

We apply the above-mentioned method to a restricted interval of altitudes. Its lower limit is determined by the time of observation such that the sun is too close to the horizon in the period of observations. The upper limit is determined by the fact that the cloud is scattered more rapidly the more diffusion begins to take place. We applied the method in the range of altitudes from 250 to 600 km, i.e., we cover almost the entire ionosphere. The latest data on the determination of the density of the atmosphere above 200 km are shown in Fig. 2.

Finally, we note that the method of the sodium cloud makes it possible to determine the temperature of the atmosphere by the Doppler width of emission lines. Next time we shall report such

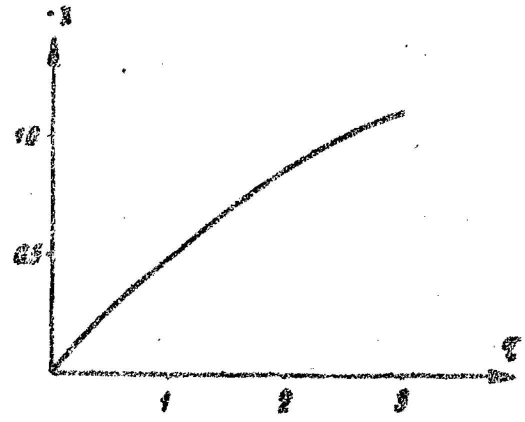

Fig. 1. Shift of the region of maximum concentration in dimensionless coordinates $x = az$, $\tau = D_0 a^2 t$.

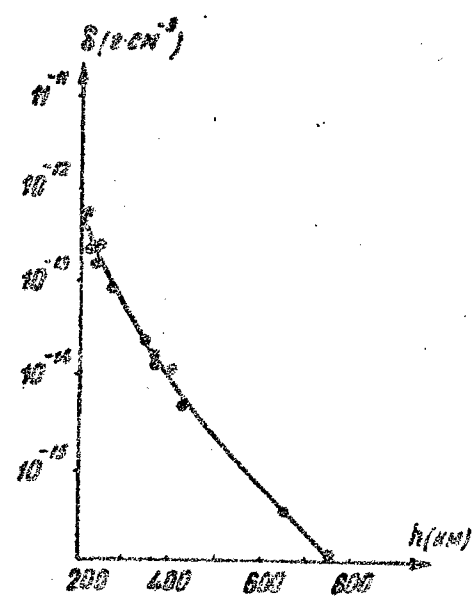

Fig. 2. Density distribution of neutral particles δ as a function of the altitude h in the ionospheric region.

observations.

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named for Sternberg

FLUCTUATIONS OF AMPLITUDE AND PHASE OF A WAVE PROPAGATED IN AN INHOMOGENEOUS ABSORBING MEDIUM

pp. 1010-1012 N. G. Denisov and L. N. Polyanin

In the calculations of fluctuations of the parameters of a wave passing through a layer with random inhomogeneities, the absorption of the waves in the layer is usually not taken into account. In the present note, such a calculation is carried out with account of absorption. Conditions are obtained, for the satisfaction of which the effective damping on the fluctuation of the parameters of the wave is unimportant.

We shall assume that the mean value of the complex dielectric constant ε' in the layer depends on z. In such a case the propagation of a scalar wave will be described by the equation

$$\Delta E + k_0^2 \left| \varepsilon'(z) + \Delta\varepsilon'(x, y, z) \right| E = 0, \qquad (1)$$

where $\Delta\varepsilon'(x, y, z)$ is the random change of the complex dielectric constant ($|\Delta\varepsilon'(x, y, z)| \ll |\varepsilon'(z)|$), $k_0 = \omega/c$ (ω is the frequency of the wave, c is the speed of light). If the regular inhomogeneous layer is sufficiently smooth, and the conditions for the applicability of the method of smooth perturbations are satisfied, in which the solution is sought in the form $E = \exp[\Phi_0(x, y, z) + \Phi_1(x, y, z) + \ldots]$,

then, taking as a zero approximation a solution describing the wave which is propagated along the z axis,

$$\Phi_0 = -ik_0\int_0^z \sqrt{\varepsilon'}\,dz - \frac{1}{4}\int_0^z \frac{d\varepsilon'}{\varepsilon'} + \ln A_0, \qquad (2)$$

we get for the function Φ_1 the solution [1]

$$\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_1}{\partial y^2} - 2ik_0\sqrt{\varepsilon'(z)}\,\frac{\partial\Phi_1}{\partial z} + k_0^2\,\Delta\varepsilon'(x, y, z) = 0. \qquad (3)$$

The time dependence is chosen in the form $e^{i\omega t}$.

We expand the random function in (3) in a Fourier integral in the variables x, y. Then we can obtain the following formula for the spectrum of the function Φ_1 [1]:

$$\varphi(\varkappa_1, \varkappa_2, z) = \frac{ik_0}{2}\int_0^{L_0} f(\varkappa_1, \varkappa_2, z)\exp\left\{i\frac{\varkappa^2}{2k_0}[v(L_1) - v(z)]\right\}\frac{dz}{\sqrt{\varepsilon'}}, \qquad (4)$$

where

$$\varphi(\varkappa_1, \varkappa_2, z) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{+\infty}\Phi_1 e^{-i(\varkappa_1 x+\varkappa_2 y)}dxdy; \quad \varkappa^2 = \varkappa_1^2 + \varkappa_2^2;$$

$$f(\varkappa_1, \varkappa_2, z) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{+\infty}\Delta\varepsilon'(x, y, z)\,e^{-i(\varkappa_1 x+\varkappa_2 y)}\,dxdy; \quad v(z) = \int_0^z \frac{dz}{\sqrt{\varepsilon'}} \qquad (5)$$

It is assumed that the observation point $z = L_1$ is outside the region where there are random inhomogeneities ($L_1 \geq L_0$). From the solution of (4), we can find the spectra of the correlation functions of the phase lead $S_1 = \mathrm{Im}\,\Phi_1$ and of the level $A_1 = \mathrm{Re}\,\Phi_1$. We first introduce some notation. Let us write the fluctuation departure of

the complex dielectric constant in the form

$$\frac{\Delta\varepsilon'(x, y, z)}{\sqrt{\varepsilon'}} = \frac{\Delta\varepsilon_1(x, y, z) - i\,\Delta\varepsilon_2(x, y, z)}{n - i\chi} = B e^{i\varphi_1}\,\Delta N(x, y, z), \tag{6}$$

where $\Delta N(x, y, z)$ is the fluctuation change of the particle density, n is the index of refraction and χ is the absorption coefficient of the medium. For plasma,

$$\varepsilon' = 1 - \frac{4\pi e^2 N}{m\omega^2(1 - is)} \qquad (s = \nu_{eff}/\omega)$$

and

$$B = \frac{4\pi e^2}{m\omega^2\sqrt{1+s^2}\,|\sqrt{\varepsilon'}|}; \qquad \varphi_1 = \operatorname{arc\,tg} s + \operatorname{arc\,tg}\frac{\chi}{n}. \tag{7}$$

Let us also write the function $i[v(L_1) - v(z)]\alpha^2/2k_0$ in the form:

$$i\frac{\alpha^2}{2k_0}[v(L_1) - v(z)] = i\frac{\alpha^2}{2k_0}\int_z^{L_1}\frac{n\,dz}{n^2+\chi^2} + \frac{\alpha^2}{2k_0}\int_z^{L_1}\frac{\chi\,dz}{n^2+\chi^2} = i\varphi_2(z) + \delta(z, L_1). \tag{8}$$

We can now write the real and imaginary parts of Eq. (4) in the following way:

$$\begin{aligned} \operatorname{Re}\varphi &= \frac{k_0}{2}\int_0^{L_0} B(z) f_N(\alpha_1, \alpha_2, z) e^{-\delta(z)} \sin(\varphi_2 + \varphi_1)\,dz; \\ \operatorname{Im}\varphi &= \frac{k_0}{2}\int_0^{L_0} B(z) f_N(\alpha_1, \alpha_2, z) e^{-\delta(z)} \cos(\varphi_2 + \varphi_1)\,dz, \end{aligned} \tag{9}$$

where

$$f_N(\alpha_1, \alpha_2, z) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\Delta N(x, y, z)\, e^{-i(\alpha_1 x + \alpha_2 y)}\,dx\,dy.$$

If we assume that the change in amplitude and phase of the field as a

result of absorption is negligibly small at distances of the order of the scale of the random inhomogeneities l, then we can obtain the following formula for the level spectra and phase lead, in the same way as in the case of a medium without absorption [2]:

$$F_{A \atop S} = \frac{k_0^2}{2}\int_0^{L_1} B^2(z)\,\overline{(\Delta N)^2}\, e^{2\delta(z)}\, \frac{\sin^2}{\cos^2}\{\alpha_2(z)+\alpha_1(z)\}\, dz \int_0^{\infty} F_N(\alpha_1, \alpha_2, \zeta)\, d\zeta, \tag{10}$$

where $F_N(\alpha_1, \alpha_2, \zeta)$ is the correlation spectrum of the function $\overline{\Delta N_1}\ \overline{\Delta N_2(\Delta N)^2}$,

$$\alpha_1 = \operatorname{arc\,tg}\frac{\nu_{eff}}{\omega} + \operatorname{arc\,tg}\frac{\chi}{n}; \quad \alpha_2 = \frac{\varkappa^2}{2k_0}\int_z^{L_1}\frac{n\,dz}{n^2+\chi^2};$$

$$\delta(z) = \frac{\varkappa^2}{2k_0}\int_z^{L_1}\frac{\chi\,dz}{n^2+\chi^2}.$$

We now calculate the spectrum F_Φ of the correlation function of the complex phase $R_\Phi(\xi, \eta) = \overline{\Phi_1(x, y)\,\Phi_1(x+\xi, y+\eta)} = R_A + R_S$:

$$F_\Phi = F_A + F_S \quad \frac{k_0^2}{2}\int_0^{L_0} B^2(z)\,\overline{(\Delta N)^2}\, e^{2\delta(z)}\, dz \int_0^{\infty} F_N(\alpha_1, \alpha_2, \zeta)\, d\zeta. \tag{11}$$

The function F_Φ is closely connected with the angular energy spectrum of the scattered field [1]. Since $\delta(z) - (\varkappa^2/2k_0)$ $\int_z^{L_1}\chi\,dz/(n^2+\chi^2)$, then we can introduce in Eq. (11) the factor $\exp\left[-(\alpha^2/k_0)\int_{L_0}^{L}\chi\, dz/(n^2+\chi^2)\right]$ which determines the change in the intensity of the components of the angular spectrum as a function of distance when there are no random inhomogeneities

$(z > L_0)$. If the absorption in the last part of the layer is equal to zero, then the angular spectrum naturally does not depend on the distance $L_1 - L_0$.

We compute the correlation function of the complex phase from the formula

$$R_\Phi(\xi, \eta) = \iint_{-\infty}^{\infty} F_\Phi \, e^{+i(\varkappa_1 \xi + \varkappa_2 \eta)} \, d\varkappa_1 \, d\varkappa_2.$$

We first write down

$$\delta(z) = \frac{\varkappa^2}{2k_0} \int_z^{L_1} \frac{\chi \, dz}{n^2 + \chi^2} = a \varkappa^2. \tag{12}$$

We then obtain the following on the basis of Eq. (11) and the theorem on the spectrum:

$$R_\Phi(\xi, \eta) = \frac{k_0^2}{8\pi a} \int_0^{L_0} B^2(z) \overline{(\Delta N)^2} \iint_{-\infty}^{+\infty} e^{-(x^2+y^2)/4a} \, dx dy \int_0^{\infty} \rho_N(\xi - x, \eta - y, \zeta) \, d\zeta, \tag{13}$$

where

$$\rho_N(\xi, \eta, \zeta) = \overline{\Delta N(x, y, z) \Delta N(x + \xi, y + \eta, z + \zeta)} / \overline{(\Delta N)^2}.$$

If the absorption in the medium vanishes, then $a \to 0$ and the function $(1/4\pi a)e^{-(x^2+y^2)/4a}$ has its own limiting delta-function. In this case, it is easy to compute the double integral (13) and obtain a formula for the medium without absorption[1]. For small a, the double integral in (13) can be computed approximately by expanding in a series the function $\rho_N(\xi - x, \eta - y, \zeta)$. Throwing away terms proportional to a^2, we find an equation which is valid for small absorption:

$$R_\Phi(\xi, \eta) = \frac{k_0^2}{2} \int_0^{L_1} B^2(z) \overline{(\Delta N)^2} \left[\int_0^\infty \rho_N(\xi, \eta, \zeta)\, d\zeta + a \int_0^\infty \nabla^2 \rho_N(\xi, \eta, \zeta)\, d\zeta \right] dz. \quad (14)$$

The second term in the brackets is much less than the first if $a/l^2 \ll 1$ throughout the range of values of z (l is the scale of random inhomogeneities). Making use of Eq. (12), we write down this inequality in the form:

$$\frac{1}{2k_0 l^2} \int_0^{L_1} \frac{\chi\, dz}{n^2 + \chi^2} = \frac{\lambda_0 L_1}{4\pi l^2} \frac{1}{L_1} \int_0^{L_1} \frac{\chi\, dz}{n^2 + \chi^2} \ll 1. \quad (15)$$

Thus upon satisfaction of the condition (15) it is not necessary in calculations of the fluctuations of the amplitude and phase to take the absorption into consideration. Hence it follows that if $\chi/(n^2 + \chi^2)$ is not large, the absorption need not be considered for $\lambda_0 L_1/l^2 \ll 1$, i.e., in the range of geometric optics.

For the ionosphere, over an interval of short waves, the inequality $\nu_{eff}/\omega \ll 1$ is usually satisfied. Here $\chi/(n^2 + \chi^2) \sim \frac{\nu_{eff}}{\omega}$, while $L_1^{-1} \int_0^{L_1} (\nu_{eff}/\omega)\, dz = \overline{\nu}_{eff}/\omega$, where $\overline{\nu}_{eff}$ is the mean value of the effective number of collisions in the layer. We consider scattering of radio waves passing through the upper ionosphere ($l \sim 10^3$ m, $\lambda_0 = 3$ m, $L_1 \sim$ 300 km, $\nu_{eff} \sim 10^3$/sec). In this case, the parameter $\lambda_0 L_1/l^2 \sim 1$, $\nu_{eff}/\omega \ll 1$ and the condition (13) is well satisfied. For the E layer, $l \sim 100$ m, $\lambda_0 = 30$ m ($\omega = 2\pi \times 10^7$/sec), $L_1 = 200$ km and the ratio $\lambda_0 L_1/l^2 \sim 500$, while $\nu_{eff}/\omega \sim 1/300$ ($\nu_{eff} \simeq 3 \times 10^5$/sec). The latter value can generally be

increased by reason of the fact that the scattered wave passes through the D layer where the number of collisions $\nu_{eff} \sim 10^6 \sim 10^7$ /sec. Thus the condition can be violated at frequencies $\omega < 2\pi \times 10^7$/sec, and in the calculation of scattering in the E layer in this frequency range it is necessary to take the absorption into account.

Finally, we note that in the calculation which has been carried out it is easy to consider also the damping of the average field brought about by the same scattering effects [3].

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ONE POSSIBILITY OF CREATING A PERIODICALLY INHOMOGENEOUS DIELECTRIC

S. S. Dukhin pages 1013-1014

In reference [1], the possibility was investigated of the creation of a periodically inhomogeneous dielectric structure by means of the generation of a standing sound wave in a liquid dielectric. Use of ultrasound for creating periodic structures is in particular convenient for investigation purposes since this method makes it possible to vary the period of the structure in a given arrangement over wide limits. However, this method in the form in which it was used in [1] brings about too small a change in the dielectric constant. The data given in [1] make it possible to calculate the relative change of density, which in this case amounts to 5×10^{-5}, in view of the small compressibility of the liquid. Naturally, the changes achieved for the parameters of the electromagnetic wave are very small, in view of the small relative change in the density and, correspondingly, in the dielectric constant [1].

The value of the ultrasonic method of obtaining periodic structures should increase sharply in its application to disperse systems, for example, to suspensions or to aerosols. In a standing sound wave, the suspended particles have a tendency to concentrate

either at the loops or at the nodes, as a result of which a periodic structure appears whose change in dielectric constant, for a sufficiently high concentration of particles, approaches a value equal to the difference of the dielectric constants of the medium and the particles. Usually, the appearance of periodicity in the distribution of dispersed particles is connected with the so-called "sound pressure," under the action of which the particles drift to the nearest node or in some cases to the nearest loop. The King formula [2] is widely used here for the sound pressure experienced by a small sphere in a standing sound wave:

$$F_{3B} = \frac{5}{2} km' (v - u)^2 f \sin (2kx), \tag{1}$$

where $k = 2\pi/\lambda$, λ is the wavelength, m' is the mass of the displaced particle of the medium, v and u are the amplitudes of the velocities of the particles and the medium, f is a factor of the order of unity, which depends on the ratio of the density of the particle and that of the medium. For derivation of this formula, the viscosity of the medium was not taken into account in [2], as a result of which, it is not possible to apply it to the very interesting case of very small particles, whose vibratory motion relative to the medium is characterized by a Reynolds number less than unity.

In this case the action of the sound wave on the particle is due essentially to viscous forces, and for a study of the drift of the particles, it is sufficient to consider the equation of vibratory

motion of the particles [3] in second approximation (which is equivalent to account of the anharmonicity of the vibrations):

$$\left(m + \frac{m'}{2}\right) \dot{v} + 6\pi\eta r \left[v + \frac{r}{\sqrt{\pi\nu}} \int_{-\infty}^{t} \frac{\dot{v}(\tau)\, d\tau}{\sqrt{t-\tau}}\right] =$$

$$= \frac{3}{2} m\dot{u} + 6\pi\eta r \left[u + \frac{r}{\sqrt{\pi\nu}} \int_{-\infty}^{t} \frac{\dot{u}(\tau)\, d\tau}{\sqrt{t-\tau}}\right]. \qquad (2)$$

where m is the mass of the particle, η and ν are the viscosity and kinematic viscosity of the medium, r is the radius of the particle, t is the time, $v = \dot{x}$ is the velocity of the particle, $u = A \sin(kx) \cdot \sin(\omega t)$ is the velocity distribution in the standing wave.

We note that Eq. (2) describes the motion in a rapidly oscillating field [4], which makes it possible to use the method of Kapitza [5]. Since the amplitude of the vibrations A is much smaller than the wavelength λ, one can consider x on the right hand side of (2) to be a constant, during a single period ($x = x_0$). Solution of Eq. (2) in this approximation determines the harmonic vibration of the particle relative to the point $x = x_0$:

$$x - x_0 = A' \sin(kx_0) \cos(\omega t + \psi), \qquad (3)$$

where $A' = A\left(1 + 3\beta + \frac{9}{2}\beta^2 + \frac{9}{2}\beta^3 + \frac{9}{4}\beta^4\right)^{1/2} \left[\frac{4}{9}\left(\frac{m}{m'} + \frac{1}{2}\right)^2 + \right.$

$$\left. + 2\beta\left(\frac{m}{m'} + \frac{1}{2}\right) + \frac{9}{2}\beta^2 + \frac{9}{2}\beta^3 + \frac{9}{4}\beta^4\right]^{-1/2};$$

$$\operatorname{tg}\psi = (1+\beta)\beta(1 - m/m')\left[\frac{m}{m'}\left(\frac{2}{3} + \beta\right) + \frac{1}{3} + 2\beta + \frac{9}{2}\beta^2 + \frac{9}{2}\beta^3 + \frac{9}{4}\beta^4\right]^{-1};$$

$$\beta = \frac{1}{r}\sqrt{\frac{2\nu}{\omega}}.$$

The right hand side of (2) can be considered as a quasi-periodic force with slowly changing amplitude, under the action of which the particle vibrates and simultaneously (in the mean) is displaced after each period. Since only those displacements whose successive sum describes the drift of the particles are of interest to us, we average Eq. (2) over the period, first expanding sin (kx) in a series relative to the point $x = x_0$: $\sin(kx) = \sin(kx_0) + k\cos(kx_0)(x - x_0)$, where we substitute (3) for $x - x_0$. The equation which is obtained as the result of averaging makes it possible to find the trajectory of the particle $x_0(t)$ averaged over a single period:

$$\left(m + \frac{m'}{2}\right)\ddot{x}_0 + 6\pi\eta r\left[\dot{x}_0 + \frac{r}{\sqrt{\pi\nu}}\int_{-\infty}^{t}\frac{\ddot{x}_0(\tau)\,d\tau}{\sqrt{t-\tau}}\right] = F(x_0), \tag{4}$$

where

$$F(x_0) = \frac{k}{4}A'A\,\omega^2 m\left[\sin\psi\,\frac{9}{4}(\beta^2 + \beta) - \cos\psi\left(\frac{3}{2} + \frac{9}{4}\beta\right)\right]\sin(2kx_0). \tag{5}$$

In the case under consideration, which is characterized by a Reynolds number less than unity, one can show that the first and third components in (4) are small in comparison with the second, whence it follows that

$$6\pi\eta r\dot{x}_0 = F(x_0). \tag{6}$$

The right hand side in Eq. (6), which has the meaning of a force averaged over the period in the case of an aerosol when $m/m' \sim 10^3$, and for not too high frequencies ($\beta^2 \gtrsim m/m'$, which corresponds to almost complete entrainment of the particle by the

vibrations of the medium) exceeds by several orders of magnitude the sound pressure computed by Eq. (1). This result is very important, since it reflects the possibility of the gathering of very fine particles into nodal planes. In particular, on the basis of Eq. (6), one can obtain for the time of accumulation Θ a quantity of the order of one second (λ = 3 cm, r = 1 micron, sound energy density Ω = 100 erg-cm^{-3}), while, using Eq. (1), we obtain: $\Theta \sim 10^3$ sec.

In order to prevent the precipitation of particles under the action of gravity, it is expedient to orient the sound beam vertically. Then the stationary distribution of particles close to the nodal plane that arises in a time of the order Θ is determined by the condition of the compensation of the weight of the particle by the force $F(x_0)$:

$$mg = F(x_0). \tag{7}$$

For example, it then follows that for λ = 3 cm and Ω = 100 erg-cm^{-3}, particles with radius r less than one micron are distributed over a range $\Delta x \sim 3 \times 10^{-2}$ cm close to the node.

The estimate given in the well-known formula of Einstein for the mean square displacement under thermal motion shows that, at least for the dimensions of particles greater than 0.1 micron, the thermal motion cannot prevent the preferential concentration of particles close to the nodes.

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THE TENSOR OF THE EFFECTIVE DIELECTRIC CONSTANT IN A MEDIUM WITH RANDOM INHOMOGENEITIES

F. G. Bass Pages 1015-1016

The following equations were obtained in references [1, 2] for the mean electric field $\bar{E}$ and the fluctuation electric field ξ:

$$\begin{gathered} \operatorname{rot}\operatorname{rot}\bar{\boldsymbol{E}} - k^2\left(\bar{\varepsilon}\,\bar{\boldsymbol{E}} + \overline{\boldsymbol{\xi}\,\delta\varepsilon}\right) = 0; \\ \operatorname{div}\left(\bar{\varepsilon}\,\bar{\boldsymbol{E}} + \overline{\boldsymbol{\xi}\,\delta\varepsilon}\right) = 0; \end{gathered} \tag{1}$$

$$\begin{gathered} \operatorname{rot}\operatorname{rot}\boldsymbol{\xi} - k^2\bar{\varepsilon}\boldsymbol{\xi} = k^2\left(\delta\varepsilon\,\bar{\boldsymbol{E}} + \boldsymbol{\xi}\,\delta\varepsilon - \overline{\delta\varepsilon\,\boldsymbol{\xi}}\right); \\ \operatorname{div}\left(\bar{\varepsilon}\,\boldsymbol{\xi} + \delta\varepsilon\,\boldsymbol{\xi} - \overline{\delta\varepsilon\,\boldsymbol{\xi}} + \delta\varepsilon\,\bar{\boldsymbol{E}}\right) = 0. \end{gathered} \tag{2}$$

Here $k = \omega/c$, $\bar{\varepsilon}$ is the mean value of the dielectric constant ε, $\delta\varepsilon$ is the fluctuation component of the dielectric constant, $\varepsilon = \bar{\varepsilon} + \delta\varepsilon$ (the bar denotes the statistical average).

If $\delta\varepsilon$ is small in comparison with $\bar{\varepsilon}$, one can neglect the terms $\delta\varepsilon\xi$ and $\overline{\delta\varepsilon\xi}$ in Eqs. (2) in comparison with $\bar{\varepsilon}\xi$, and then find ξ. Substituting the value of ξ from (2) in (1), and carrying out the averaging, we finally obtain an equation for the components of the electric field [1, 2]:

$$\left[\left(\Delta + \bar{\varepsilon}k^2\right)\delta_{ik} - \frac{\partial^2}{\partial x_i \partial x_k}\right]\bar{E}_k(\boldsymbol{r}) +$$

$$\ldots \frac{k^2 \overline{\delta\varepsilon^2}}{4\pi} \int dr' \, \frac{e^{ik\sqrt{\bar{\varepsilon}}\,|r-r'|}}{|r-r'|} \left(\delta_{ik} + \frac{1}{k^2\bar{\varepsilon}} \frac{\partial^2}{\partial x_i \partial x_k}\right) \left(W(r-r')\, E_k(r')\right) = 0, \quad (3)$$

where $W(\underset{\sim}{r} - \underset{\sim}{r}') = \dfrac{\overline{\delta\varepsilon(\underset{\sim}{r})\, \delta\varepsilon(\underset{\sim}{r}')}}{\overline{\delta\varepsilon^2}}$ is the correlation coefficient for the fluctuations of the dielectric constant, which we shall consider to be an even function of its argument.

Let us consider the case in which the electric field changes at distances considerably larger than those at which the correlation coefficient falls off appreciably. In this approximation, $\overline{E}_k(r')$ can be taken out from under the integral in Eq. (3) and we can rewrite the integral in the following form:

$$\left(\Delta\delta_{ik} + k^2 \varepsilon'_{ik} - \frac{\partial^2}{\partial x_i \partial x_k}\right) \overline{E}_k(r) = 0, \quad (4)$$

where the effective complex tensor of the dielectric constant is determined by the equation:

$$\varepsilon'_{ik} = \bar{\varepsilon}\,\delta_{ik} + \frac{k^2 \overline{\delta\varepsilon^2}}{4\pi} \int d\rho \, \frac{e^{ik\sqrt{\bar{\varepsilon}}\,\rho}}{\rho} \left(\delta_{ik} + \frac{1}{k^2\bar{\varepsilon}} \frac{\partial^2}{\partial \rho_i \partial \rho_k}\right) W(\rho). \quad (5)$$

If the correlation coefficient depends only on the modulus of the vector $\underset{\sim}{\rho}$, then the tensor (5) degenerates to a scalar and we again arrive at the results of reference [2], only with this difference that in our case $\bar{\varepsilon}$ is not equal to unity.

In the same case, when $W(\underset{\sim}{\rho})$ again depends on ρ_x, ρ_y, ρ_z, that is, the inhomogeneities are anisotropic, ε'_{ik} is a symmetric

complex tensor.

The average electric field in a medium with random anisotropic inhomogeneities is described by the same equations as in a crystal; consequently, in such a medium two waves can propagate with different phase velocities [3].

If the corresponding integrals converge, then one can expand the exponent in the integrand of Eq. (3) in a series in $k\sqrt{\bar{\varepsilon}}\,\rho$. In this case, the following expression is obtained for ε'_{ik}:

$$\varepsilon'_{ik} = \bar{\varepsilon}\,\delta_{ik} + \frac{\overline{\delta\varepsilon^2}}{4\pi\bar{\varepsilon}} \int \frac{d\rho}{\rho} \frac{\partial^2 W}{\partial\rho_i\,\partial\rho_k} + \frac{ik^3\overline{\delta\varepsilon^2}\sqrt{\bar{\varepsilon}}}{4\pi} \left[\int d\rho\; W(\rho)\,\delta_{ik} - \frac{1}{6} \int d\rho\,\rho^2 \frac{\partial^2 W}{\partial\rho_i\,\partial\rho_k} \right]. \qquad (6)$$

We consider the case in which $\delta\varepsilon$ depends on a single coordinate, for example, on x. Such a correlation coefficient depends only on ρ_x. We cannot make use of Eq. (6) directly, since the integrals entering into it diverge in this case. However, one can obtain an answer if integration is carried out in Eq. (5) over ρ_y and ρ_z, and the resulting expression expanded in a series in powers of $k\sqrt{\bar{\varepsilon}}$. As a result we obtain:

$$\varepsilon'_{xx} = \bar{\varepsilon} - \frac{\overline{\delta\varepsilon^2}}{\bar{\varepsilon}}; \quad \varepsilon'_{ik} \left(\bar{\varepsilon} + \frac{ikl\,\overline{\delta\varepsilon^2}}{\sqrt{\bar{\varepsilon}}} \right) \delta_{ik}, \qquad (7)$$

where $l = \int_0^\infty W(\rho_x)\,d\rho_x$; $i \neq x$ or $k \neq x$.

We note that the expression for the effective tensor of the dielectric constant can be obtained in the case of a dependence of

the fluctuation $\delta\varepsilon$ on a single **coordinate** without assumptions on the smallness of $\delta\varepsilon$, if we limit ourselves to the zero approximation in k. Actually, neglecting terms of order of $k^2\bar{\varepsilon}$ in Eqs. (2), we see that the field in zero approximation can be regarded as a potential field and we can limit ourselves to only the second equation of the set (2). Furthermore, in the second of the equations of the system (2), it is possible to neglect ξ_y and ξ_z, since these quantities are of first or higher order in $k\sqrt{\bar{\varepsilon}}$. After the simplifications mentioned, we get the following from the second equation of the system (2):

$$\xi_x = \frac{\overline{\delta\varepsilon\xi_x}}{\varepsilon} - \frac{\varepsilon - \bar{\varepsilon}}{\varepsilon}\,\bar{E}_x. \tag{8}$$

Averaging Eq. (8), and assuming that $\bar{\xi}_x = 0$, we have:

$$\overline{\delta\varepsilon\xi_x} = \left[\left(\overline{\frac{1}{\varepsilon}}\right)^{-1} - \bar{\varepsilon}\right]\bar{E}_x. \tag{9}$$

It then follows that the tensor ε'_{ik} has the form:

$$\varepsilon'_{xx} = \left(\overline{\frac{1}{\varepsilon}}\right)^{-1}; \quad \varepsilon'_{ik} = \bar{\varepsilon}\,\delta_{ik}, \tag{10}$$

where $i \neq x$ or $k \neq x$. Of course, for small $\delta\varepsilon$, Eq. (10) goes over into (7) with accuracy up to terms of order $k\sqrt{\bar{\varepsilon}}$.

We note that Eq. (10) coincides in form with the equation obtained by another method by Faǐnberg and Khizhnyak [4] for a periodically inhomogeneous dielectric. We also note that, as is seen from (3) and (7), the anisotropy of the effective tensor of the dielectric constant will be especially clearly demonstrated in a dispersing medium at frequencies close to the roots of $\bar{\varepsilon}$, for

example, in a plasma at frequencies close to the Langmuir frequency.

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APPLICATION OF ULTRASOUND TO THE CREATION OF ALTERNATING FIELDS AND PERIODIC STRUCTURES

I. M. Vigdorchik Pages 1016-1017

In searches for new methods of generation of electromagnetic waves, attention is turned to the possibilities which arise in connection with the use of ultrasonic techniques for the purposes of creation of both variable fields and periodic structures. With the use of ultrasound one can appreciably reduce the period of a delay system and can make more frequent alternation of variable fields, and at the same time obtain the possibility of transition to shorter length radio waves.

One of the ways of using ultrasound for the creation of a periodic structure is the modulation of the plasma of a gas discharge by a standing sound wave [1]. In this case a difference arises in the pressure between neighboring antinodes of the standing waves, and, consequently, a corresponding difference of the dielectric constant $\Delta\varepsilon$, equal to $(\varepsilon - 1)\ \Delta P/P$. In the well-known approximation, one can consider such a system as a set of alternating layers of two dielectrics.

Passing a beam of uniformly moving electrons through such a medium for a corresponding density and velocity of electrons

one can expect the appearance of electromagnetic radiation, whose frequency will be determined by the periodicity of the structure of the plasma of the gas discharge. In other words, in the case described there appears one of the varieties of parametric resonance, called by Fainberg and Khizhnyak the Cherenkov parametric effect [2]. They showed that in the alternating layers of two dielectrics a redistribution of the total energy loss of the moving electrons takes place. The polarization losses in this case become much less than the energy loss in the Cherenkov radiation and consequently such a system will generate electromagnetic waves. Observing the known relations between the ultrasonic wavelengths and the gas density in the gaseous space, one can expect to obtain very short electromagnetic waves.

The difficulties in the operation of such a method of generation of electromagnetic waves are brought about by the fact that the scattering of the beam of electrons increases with increase in the gas pressure and a decrease in pressure leads to an increase in ultrasonic absorption in the gas. However, the corresponding choice of the initial velocity of the electrons, frequency of the ultrasound and gas pressure make it possible to find a satisfactory solution of the problem.

We can demonstrate one of the possibilities of obtaining a variable field by means of ultrasound through setting up standing ultrasonic waves in a piezoelectric specimen. If a standing

ultrasonic wave is generated in a piezoelectric sample of thickness b (Fig. 1), located between metallic plates M, then an electric field which is variable in magnitude and direction arises in the gap a because of the inverse piezoelectric effect along the plates. The value of this field at the antinode of the standing wave will be equal to [3]

$$E_0 = d_{11} \frac{4\pi}{1 + \varepsilon a/b} Y_y,$$

where d_{ij} is the piezoelectric modulus, ε is the dielectric constant, Y_y is a component of the elastic deformation tensor.

If one can obtain pressures of the order of 10^6 dynes/cm^2 with a sound intensity of the order of tenths of watts per centimeter, we can obtain an electric field intensity of the order of 1000 volts/cm by using a dielectric with a piezoelectric constant equal to 10^{-6}. An electron falling into such a variable electric field and moving with some initial velocity will carry out harmonic vibrations and radiate monochromatic electromagnetic waves. The frequency of this radiation will depend on the velocity of motion of the electron, the wavelength of the ultrasound and the dielectric constant of the piezoelectric material.

The possibility of use of an oscillator moving above a dielectric for the production of electromagnetic waves was pointed out by Ginzburg in 1947 [4].

The examples given for the use of ultrasound for the production of a periodic structure and variable fields are not unique.

However, they show the possibilities which are not currently employed and point out new perspectives for us.

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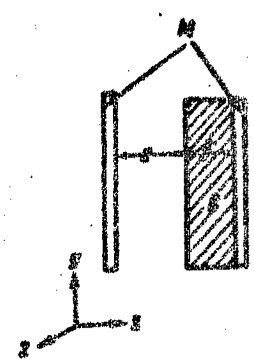

Fig. 1

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